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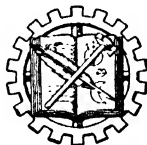
A FIRST COURSE IN MATHEMATICS

*FOR STUDENTS OF ENGINEERING
AND THE PHYSICAL SCIENCES*

BY

EDWARD BAKER
Newark College of Engineering

THIRD PRINTING



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PREFACE

The general plan of this book is that of proceeding by easy stages from less difficult to more difficult subjects. In following this pattern, the study of analytic geometry is postponed to the latter third of the book, since the locus concept, though simple, is by no means easy to grasp. This consideration is, of course, less important in the case of exceptionally capable or well prepared groups.

In a field so thoroughly worked and reworked, originality of treatment is not to be expected. The steadily increasing importance of vector concepts and methods, in pure as well as in applied mathematics, has, however, seemed to justify a greater degree of emphasis upon vector ideas than is customary in an elementary textbook. This has resulted in a simpler and more direct approach to certain topics, such as the distance from a point to a line.

I have somewhat reluctantly included a chapter on the calculus. Most of my colleagues will undoubtedly prefer to introduce this subject in their own way, and at their own time. I most heartily concur with those who maintain that some calculus should be absorbed during the first year of college mathematics; but how and when this material should be introduced is not a question to which a categorical answer is possible. In selecting the method of presentation for the chapter on derivatives and integrals, I have been guided by the needs of those students who require some knowledge of calculus for a sophomore course in physics or in engineering mechanics, before the regular course in calculus is taken.

It is pleasant to recall the interest and the cordial spirit of cooperation which my colleagues at the Newark College of Engineering have manifested in the preparation of the book. Professor J. H. Fithian has most generously made available to me the materials and plans which he has developed over a period of many years. He has, moreover, read the entire manuscript with care, so that almost every page has benefited by his suggestions.

Many of the problems originated with Professor P. O. Hoffmann; he also read and discussed with me many portions of the manuscript. Dr. E. C. Easton read part of the manuscript, and contributed many original problems. Professor P. Mainardi, Mr. C. Konove, and Dr. J. Frechafer

contributed problems and suggestions. Professor J. M. Robbins suggested several problems from the field of surveying. Professor F. J. Burns offered many helpful suggestions in connection with the figures in Chapter 20.

My thanks are due also to others of my friends and colleagues who have assisted me in various ways, and to the publishers, for expert and courteous help in the mechanics of bookmaking.

EDWARD BAKER.

Newark, N. J.
August 5, 1942.

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In the manufacture of this book, the publishers have observed the recommendations of the War Production Board and any variation from previous printings of the same book is the result of this effort to conserve paper and other critical materials as an aid to the war effort.

CHAPTER 1

TRIGONOMETRIC FUNCTIONS AND VECTORS

1. Vector Quantities. One of the basic concepts in applied mathematics is that of a quantity which has direction as well as magnitude. An example from ordinary life is *force*. Suppose that we wish to move a heavy box from a point P_1 to a new location P_2 , by pulling on a rope attached to the box. To achieve the desired result, it is necessary not only to apply a force sufficiently large to move the object, but also to apply it in the right direction, that is, in the line running from P_1 to P_2 . The force may be represented by a line segment AB , as shown in Figure 1. The length of AB represents the magnitude of the force. The direction of AB represents the direction of the force.

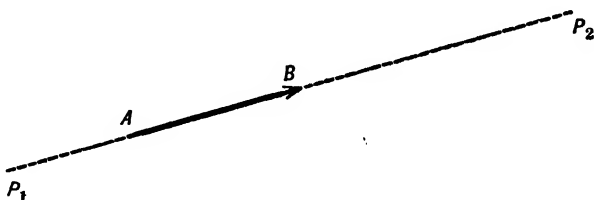


FIG. 1

The segment AB is called a line vector. Any of the vector quantities encountered in ordinary technical work may be represented by a line vector, just as has been done for a force. For our purposes, it may be assumed that the mathematical properties of all vector quantities are identical with those of line vectors.

A question that is both interesting and important naturally arises at this point: How can one know whether a certain quantity may or may not be treated as a vector? To this question the mathematician has what for him, at least, is a satisfactory answer; but unfortunately the mathematician's test is somewhat abstract, and not at all suitable for an elementary exposition. For practical purposes we may say that a vector quantity is one which satisfies the triangle law, presently to be stated.

2. Components. The projection of a vector on any line is called a component* of the vector in the direction of the line. Thus AC , in Figure 2, is said to be the horizontal component of the vector AB .

To gain some insight into the meaning of components, let us consider another familiar example of a vector quantity, namely velocity. Suppose an airplane to be flying northeast with a velocity of 200 miles per hour.

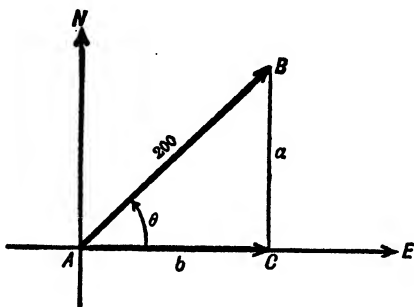


FIG. 2

The situation may be represented by a vector diagram as in Figure 2. From the theorem of Pythagoras we see that

$$a^2 + b^2 = 200^2$$

Taking $\theta = 45^\circ$, we know from elementary geometry that $a = b$, and thus

$$\begin{aligned} b = a &= \frac{200}{\sqrt{2}} \\ &= 141 \text{ approximately} \end{aligned}$$

The motion of the airplane is completely described by giving the easterly and northerly components of its velocity, which are each 141 miles per hour. That is, during every hour, the airplane advances 141 miles to the east, and also 141 miles to the north.

Any vector quantity may be completely specified by its components. The importance of this lies in the fact that most calculations having to do with vector quantities are performed by means of components. Our first problem is to consider how we may find the components of a vector.

3. The Sine and Cosine Functions. In Figure 2, let AB be any vector, and let its direction be specified by the angle θ which it makes with some

* More accurately, a rectangular component. See below, page 4.

fixed direction AE . The length of AB , and the length of its component along AE , bear a certain relation to each other, which depends upon the angle θ . That is, if AB is multiplied by a certain numerical factor, the result is the component AC . The multiplying factor is called the cosine of θ . It is obviously never greater in value than 1. The cosine function is formally defined by the equation

$$AC = AB \cos \theta \quad 1-1$$

which expresses in the terse and precise language of mathematics the ideas previously stated in words in this paragraph.

The form of this definition suggests that the value of the factor $\cos \theta$ depends only upon the size of the angle. That this is true may be seen as follows.

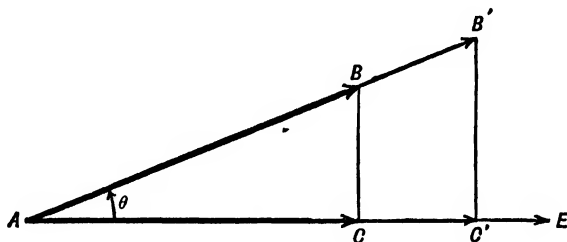


FIG. 3

In Figure 3, AB and AB' are two vectors of unequal magnitudes, both making the same angle with the fixed direction AE . The two right triangles are similar; hence AB and its component AC are in the same ratio as AB' and its component AC' . Thus the factor by which the component is calculated is independent of the magnitude of the vector. It is a function of the size of the angle of the vector, and that alone.

The sine of an angle is defined as that quantity which, multiplied by the length of a vector, gives the component in a direction perpendicular to the fixed direction. (It is understood that the vector, its component, and the line indicating the fixed direction, all lie in the same plane.) Referring to Figure 4, this statement may be written

$$CB = AB \sin \theta \quad 1-2$$

A vector may be resolved into components in various ways. The way that has been described, resulting in components perpendicular to one another, is the only one that need be considered by us. Components of

this kind (that is, mutually perpendicular) are called *rectangular*. No other kinds of components are discussed in this book.

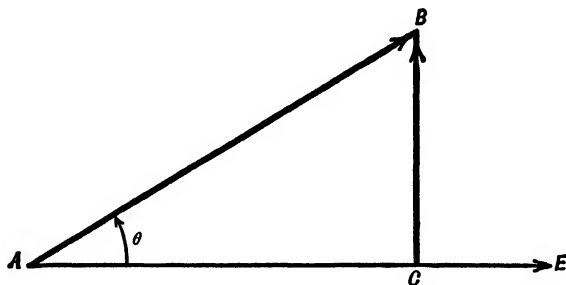


FIG. 4

In order to calculate the rectangular components of any vector we need to know the values of the cosine and the sine of the angle of the vector. In practice these values are taken from conveniently arranged tables.

Example 1. Suppose that a projectile is fired from a gun held at an angle of $17^{\circ}30'$ with the horizontal. The muzzle velocity is 2300 feet per second. The horizontal and vertical velocity components are required. (See Figure 5.)

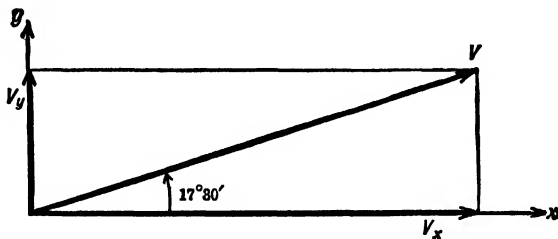


FIG. 5

From the tables, $\sin 17^{\circ}30' = 0.301$, and $\cos 17^{\circ}30' = 0.954$. (The values obtained from the tables have been rounded off to three places of decimals. See Chapter 2.) Therefore the horizontal component is

$$\begin{aligned} v_x &= 2300(0.954) \\ &= 2200 \text{ feet per second} \end{aligned}$$

The vertical component is

$$\begin{aligned} v_y &= 2300(0.301) \\ &= 690 \text{ feet per second} \end{aligned}$$

Example 2. An object weighing W lb. rests upon an inclined plane. A gravitational pull of W lb. is exerted by the earth upon the object. It is required to

find the components of this force along the plane, and at right angles to it (Figure 6).

If the plane is inclined at an angle θ to the horizontal, then by a well known theorem from geometry, the angle between the force W and its component normal

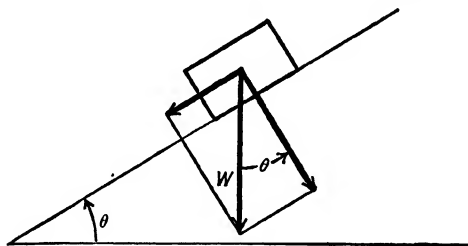


FIG. 6

(perpendicular) to the plane is likewise θ . Hence the normal component is $W \cos \theta$; the component down the plane is $W \sin \theta$.

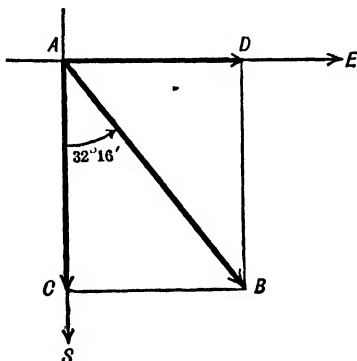


FIG. 7

Example 3. The data of this example were obtained during a survey. The surveyor's notebook shows that two points A and B are 419.2 feet apart, and that from A , point B bears South $32^\circ 16'$ East (Figure 7). The two points are at the same elevation. It is required to find the north-south distance, and also the east-west distance, from A to B .

The usual practice among surveyors is to measure the "bearing angle" from a north-south line. This line, then, gives the fixed direction.

From the tables, $\sin 32^\circ 16' = 0.5339$, and $\cos 32^\circ 16' = 0.8456$. The north-south distance between A and B is

$$419.2 \cos 32^\circ 16' = 354.5 \text{ feet}$$

The east-west distance between A and B is

$$419.2 \sin 32^\circ 16' = 223.8 \text{ feet}$$

Exercises

1. A force of 220 lb. acts in a direction making an angle of 37° with the horizontal. Find the horizontal and vertical components of the force. *Ans.* 180 lb., 130 lb.
2. A post 12 ft. long stands at a slant so that the angle made with the vertical direction is 28° . If a force of 3200 lb. is applied along the axis of the post, what are the horizontal and vertical components of the force?
3. A car moves up a plane at the rate of 18 ft. per second. The plane is inclined at an angle of 17° to the horizontal. What is the vertical component of the car's velocity? *Ans.* 5.3 ft. per sec.
4. A ship is sailing at 31 miles per hour in a direction 36° west of north. Find the northerly and westerly components of its velocity.
5. Two men lift a stone by means of ropes, which lie in the same vertical plane. One man pulls with a force of 120 lb., in a direction at 20° with the vertical. Find the horizontal and vertical components of this force. *Ans.* 41 lb., 110 lb.
6. Referring to Exercise 5, the second man pulls in a direction making an angle of 28° with the vertical. The horizontal component of his pull equals the horizontal component of the 120 lb. force exerted by the first man. Find the force exerted by the second man.
7. An electron is moving in an electric field with a velocity of 56,000 kilometers per second. The velocity vector makes an angle of 68° with the direction of the electric field. Find the components of velocity of the electron in the direction of the field, and at right angles to it.

4. The Tangent Function. We have seen that with the help of a table of sines and cosines, the components of any vector may readily be found. Of equal importance is the converse problem, namely, finding a vector and its angle when the components are known. As an illustration, suppose that an airplane is headed due east, and flying at an air speed of 200 miles per hour. The pilot is aware that the wind is blowing towards the north at an average speed of 35 miles per hour. What is his velocity with respect to the earth?

The pilot's problem is represented in Figure 8. It is evident from the diagram that the ground speed may be calculated by the theorem of Pythagoras.

$$AB^2 = 200^2 + 35^2$$

$$AB = 203 \text{ m.p.h.}$$

The unknown angle at A may be calculated with the aid of a table of sines.

$$35 = AB \sin \theta$$

$$\sin \theta = 0.172$$

$$\theta = 10^\circ$$

This kind of problem recurs so often that it is worthwhile to consider what may be the most efficient means for solving it. Another solution, which has some decided practical advantages, may be obtained in the following way.

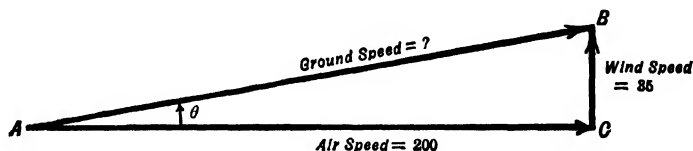


FIG. 8

We begin by defining a third trigonometric function, called the tangent function, by the equation

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad 1-3$$

The value of a fraction is unchanged if numerator and denominator are multiplied by the same quantity. Hence

$$\begin{aligned} \tan \theta &= \frac{AB \sin \theta}{AB \cos \theta} \\ &= \frac{35}{200} \\ &= 0.175 \end{aligned}$$

From a table of tangents, we find that, as before,

$$\theta = 10^\circ$$

The ground speed may now be calculated from the equation

$$\begin{aligned} 35 &= AB \sin \theta \\ AB &= \frac{35}{\sin \theta} \\ &= 203 \text{ m.p.h.} \end{aligned}$$

It will be observed that the second solution avoids the use of the Pythagorean theorem, which, though simple in principle, is awkward in computations involving large numbers.

Exercises

1. A certain town is 220 miles to the east, and 140 miles to the north of Philadelphia. What is its distance and direction from Philadelphia? *Ans.* 260 miles, N57°30'E
2. Point B is located 450 ft. to the east, and 390 ft. to the north of point A . Find the distance from A to B , and the direction of B from A .
3. The components of a vector are 820 and 630. Find the magnitude of the vector, and the angle made by it with the larger component. *Ans.* 1030, 37°30'
4. The components of a vector are 57 and 72. Find the magnitude of the vector, and the angle made by it with the larger component.
5. A man moves a heavy trunk, partly lifting and partly sliding it by pulling on a rope. He finds that less effort is required when the vertical component of the force exerted by him is one-half of the horizontal component. Find the angle made by the rope with the horizontal, and find the total force exerted if the vertical component is 75 lb.

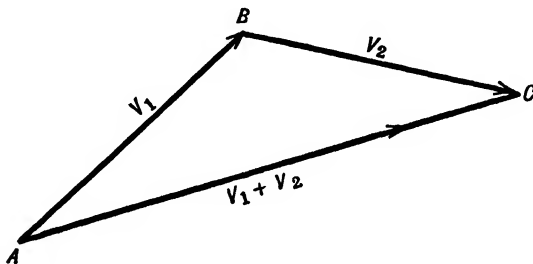


FIG. 9

5. The Triangle Law. The relation between a vector and its components is in reality a special case under the following law:

If two vectors of the same kind are in magnitude and direction represented by two sides of a triangle, then the third side of the triangle represents in magnitude and direction the vector sum of the two vectors (Figure 9).

It is understood that the triangle is so drawn that, if the first vector runs from A to B , the direction of the second vector is from B to C , as indicated by the arrowheads. The vector sum then runs from A to C .

The foregoing relation between two vectors and their sum is often expressed in an alternative form called the *parallelogram law*:

If two vectors of the same kind are in magnitude and direction represented by two adjacent sides of a parallelogram, then the vector sum of the two vectors is represented in magnitude and direction by a diagonal of the parallelogram (Figure 10).

The parallelogram must be constructed in such a way that, if the first vector runs from O to A , the second runs from O to B . The vector sum has the direction from O to C .

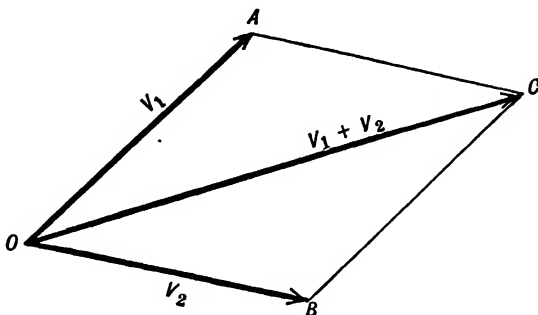


FIG. 10

6. Fixed Directions. Implicit in our treatment of vectors and their components has been the idea that it is possible to establish reference lines. It is an idea that merits closer scrutiny.

Most of our everyday activities are oriented with respect to the direction *down*, since everyone is continually subject to the earth's gravitational pull. Let us imagine an object of conveniently small size which can be placed at any desired point. The earth pulls upon this object with a force proportional to its mass. Thus we may associate with any point a force vector pointing towards the center of the earth. Considering the set of vectors associated with every point in a certain region, we arrive at the concept of a *vector field*. Such fields play an important role in modern scientific thought. Our present interest is caused by the circumstance that "fixed directions" are often selected with regard to a vector field of some kind.

Another example of a vector field is afforded by the earth's magnetic field, which can be measured in strength and direction at any point by a sliver of magnetized steel. Surveyors and explorers make use of the earth's magnetic field for a direction of reference.

7. The Complementary Relations. Modern trigonometric tables are constructed only for angles ranging from 0° to 45° . For angles in the interval from 45° to 90° , the values of sine and cosine are found by making use of the fact that *the sine of any angle is equal to the cosine of the complementary angle*.

To prove this, consider a vector OA making an angle θ with the horizontal line OX (Figure 11). The vertical component of the vector is

$$OB = OA \sin \theta$$

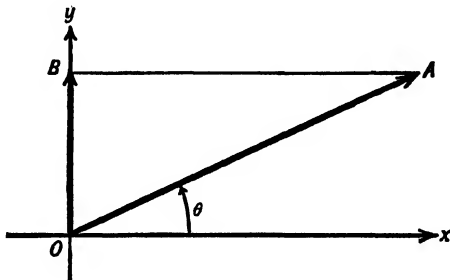


FIG. 11

Now there is no reason why we may not take the fixed direction as established by the vertical line OY , with which OA makes an angle $90^\circ - \theta$. Accordingly, we have

$$OB = OA \cos (90^\circ - \theta)$$

Hence we see that

$$\sin \theta = \cos (90^\circ - \theta) \quad 1-4$$

as was to be proved. On account of this relationship, the sine and cosine are said to be complementary functions.

8. The Cotangent Function. Let us define a new function, the cotangent, by the equation

$$\cot \theta = \frac{1}{\tan \theta} \quad 1-5$$

To justify the name, it will be shown that the tangent of any angle equals the cotangent of the complementary angle. First, we remark that, in view of equation 1-3,

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

It follows that

$$\begin{aligned} \cot (90^\circ - \theta) &= \frac{\cos (90^\circ - \theta)}{\sin (90^\circ - \theta)} \\ &= \frac{\sin \theta}{\cos \theta} \end{aligned}$$

$$\cot (90^\circ - \theta) = \tan \theta \quad 1-6$$

as was to be proved.

Exercises

1. Show that

$$\cot A \cos (90^\circ - A) = \cos A$$

for any angle A .

2. Simplify the expression

$$\frac{\sin (90^\circ - B)}{\cot B}$$

by means of the formulas of this chapter.

3. Find a value of ϕ for which

$$\cos \phi = \sin 2\phi$$

$$\text{Ans: } \phi = 30^\circ$$

4. Find a value of x for which

$$\frac{\cos x}{\sin x} = \tan 4x$$

5. By reasoning similar to that used in establishing formula 1-4, prove that

$$\cos \theta = \sin (90^\circ - \theta)$$

6. Two forces, of 65 lb. and 88 lb. respectively, act at an angle of 41° with one another. Construct to scale a triangle of forces from which the vector sum may be obtained. Obtain graphically the magnitude of the vector sum (or resultant), and measure the angle made by it with the 88-lb. force.
7. Solve Exercise 6 by constructing the parallelogram of forces.
8. Two forces in the same vertical plane act on a body. The first has a magnitude of 3 tons, and acts at an angle of 35° with the horizontal, upward to the left. The second has a magnitude of 5 tons, and acts at an angle of 25° with the horizontal, upward to the right. Construct to scale a triangle of forces, and measure the magnitude and direction of the resultant force.
9. Solve Exercise 8 by constructing the parallelogram of forces. Verify that the vertical component of the vector sum is equal to the sum of the vertical components of the given forces.

CHAPTER 2

NUMERICAL COMPUTATION. RADIAN MEASURE

9. Exact and Approximate Numbers. There are forty-eight states in the Union. There are one hundred cents in a dollar. In statements such as these, the numbers 48 and 100 are said to be exact. On the other hand, when someone speaks of the speed of a car as 60 miles per hour, we may reasonably infer that the speed lies between 55 and 65 miles per hour; or, if it is known that the figure was obtained under special test conditions, we may perhaps infer that the speed lies between 59 and 61 miles per hour. Under no circumstances may we reasonably infer that the number 60 is exact.

Most of the numbers which engineers and scientists employ are obtained directly or indirectly from measurements, typically measurements of length. Any number representing a measurement of length is an approximate number.

10. Significant Figures. In precise work, it is good practice to specify the margin of error in every number that is subject to errors in measurement. For example, the temperature at which a certain laboratory experiment is performed might be given as 62.4 ± 0.3 degrees.

Unless the margin of error is explicitly stated, or clearly indicated by the context, good practice requires that only as many significant figures be retained as are believed to be correct. For the purposes of this book, any digit from 1 to 9 will be regarded as significant. Thus the temperature in the experiment of the preceding paragraph would be given as 62 degrees. The two digits 6 and 2 are both significant, and the margin of doubt has been (roughly) indicated by the use of *two significant figures*. Obviously the use of significant figures to indicate the degree of doubt that should be attached to a measured quantity is a somewhat crude device. It has the practical advantages of being simple and convenient.

The number of significant figures in any approximate number is independent of the position of the decimal point. For example, the barometric pressure may be expressed as 756 millimeters, or as 75.6 centimeters, or as 0.756 meters. In each case, the atmospheric pressure has been given to three significant figures.

Zeros are regarded as significant unless they are used merely to fix the decimal point. For example, a number such as 0.002 830 is said to be given to four significant figures. The first three zeros are used to fix the decimal point. The final zero is not needed to fix the decimal point; its presence means that the sixth place of decimals is believed to be more precisely given by zero than by any other digit. It may be inferred that the number lies between 0.002 829 5 and 0.002 830 5. The *absolute error* is not greater than 0.000 000 5, and the *relative error* is not greater than

$$\frac{0.000\ 000\ 5}{0.002\ 830}$$

or approximately 1 part in 5000.

A difficulty arises in the case of a number such as 720. Does the final zero merely indicate the position of the decimal point, or may we infer that the units place is nearer zero than any other digit? Either interpretation is defensible. The ambiguity can be avoided by using the form $7.2(10^2)$, if two significant figures are to be understood; or $7.20(10^2)$, if three significant figures are known. This notation will be taken up later (Chapter 9). For the present, let it be agreed that an approximate number such as 720 is expressed to two significant figures only, unless the contrary is explicitly stated.

11. Rounding Off. It may happen that a number is expressed to more significant figures than may properly be used. Suppose, for example, that an electrical circuit contains two resistances connected in series, one having the value 0.21 ohms, and the other 33.7 ohms. The resistance of the combination is the sum of these values. If one of the two resistances is known only within one-tenth of an ohm, the value of the two together cannot be known more precisely. Hence the sum must be rounded off by dropping the hundredths place. This gives 33.9 ohms as the total resistance.

It is often hard to know just how many significant figures ought to be retained in expressing a number. It is sometimes helpful to use a special symbol, such as *, to represent a doubtful figure, which may be any integer from zero to nine. The sum obtained by adding any digit to * will be doubtful, and the product obtained by multiplying any digit by * will be doubtful.

Example 1. The problem of the two resistances may be worked thus:

$$\begin{array}{r} 0.21 \\ 33.7* \\ \hline 33.9* \end{array}$$

14 Numerical Computation. Radian Measure

The hundredths place is seen to be doubtful.

Example 2. Multiply 1.2 by 27.3.

Indicating doubtful digits by *, we have

$$\begin{array}{r} 1.2* \\ \underline{27.3} \\ 36* \\ 8\ 4* \\ \underline{24\ *} \\ 32.\ *** \end{array}$$

By the ordinary process of multiplication, the result is 32.76. The digits 7 and 6 are both doubtful, and therefore the answer should be given as 33.

The use of the symbol * is suggested only as an aid to the student in working out for himself the principles involved in computation with approximate numbers. It is too cumbersome to be employed by computers. For practical use, what is needed is a set of rules for computation. Because the use of significant figures expresses somewhat inexactly the margin of error of a measured number, the following rules for computation do not always lead to consistent results when a numerical problem is solved by diverse methods. However, they do suggest the guiding principles of correct computing practice, and, if reinforced by the student's own good sense, they will prove convenient and reliable.

12. Rules for Computation. (a) In rounding off a number, increase the last digit retained by 1 if the discarded figures amount to more than half a unit in the last place retained. Retain the last digit unchanged if the discarded figures amount to less than half a unit. If the discarded figures amount to just half a unit, round off to the nearest *even* number (to avoid an accumulation of errors in long computations).

Examples. 2550 rounded off becomes 2600

2450 rounded off becomes 2400

6453 rounded off to two significant figures becomes 6500

(b) In addition or subtraction, retain one column of doubtful figures. Round off the final answer.

Example.

$$\begin{array}{r} 42.\cancel{8} \\ 7316 \\ \underline{250} \\ 7608 \end{array}$$

The answer should be rounded off to 7610. In this example, the units column is doubtful, because of the number 250. Hence the tenths column is discarded immediately.

(c) In multiplication or division, determine the number of significant figures in each factor. Then find the smallest of these numbers; the final answer should be rounded off to this number of significant figures. One additional significant figure should be carried in the intermediate steps.

Example. Find the value of

$$Q = \frac{(0.234) (7500)}{0.01405}$$

The number of significant figures in the three factors is three, two, and four. The final answer must be rounded off to two significant figures. Three significant figures should be retained in the intermediate steps. Accordingly

$$\begin{aligned} Q &= \frac{(0.234) (7500)}{0.0140} \\ &= \frac{1760}{0.0140} \\ &= 126,000 \end{aligned}$$

The answer should be rounded off to 130,000.

(d) In computations performed by means of logarithms (see Chapter 9), the number of decimal places retained in the mantissa should be the same as the number of significant figures required for intermediate steps under rule (c).

Exercises

1. Give the number of significant figures in each of the following numbers:

4360, 436.0, 0.0436

2. Give the number of significant figures in each of the following numbers:

7200, 720.0, 0.0072

Ans. 2, 4, 2

3. Round off each of the following numbers to two significant figures:

13.58, 0.565, 394.7, 505

4. Round off each of the following numbers to three significant figures:

92.343, 42.752, 0.7945, 1.375

5. If the sides of a rectangle are 6.3 ft. and 9.1 ft., what is the perimeter? The area?

Ans. 30.8 ft., 57 sq. ft.

6. If the sides of a rectangle are 8.8 ft. and 5.3 ft., what is the perimeter? The area?
7. Express the following numbers in more convenient units, in such a way that non-significant figures are eliminated: 27,000 grams; 0.0271 meters; 0.02 dollars; 0.0064 liters; 17,000 milligrams. (One meter is a thousand millimeters, and one kilometer is a thousand meters; similarly for grams, milligrams, and kilograms. One liter is a thousand cubic centimeters.)
8. Give the value of $\frac{3}{17}$ expressed as a decimal, correct to four significant figures.
9. If $a = 6.728$, $b = 3.15$, and $c = 2.5$, to how many significant figures should $a - b - c$ be expressed? To how many should $a + b + c$ be expressed? $(a + b)c$?
Ans. 2, 3, 2
10. If $a = 1.94$, $b = 3.2$, and $c = 4.48$, to how many significant figures should $a + b + c$ be expressed? To how many should $a + b - c$ be expressed?
11. In an analysis the following amounts of iron were found in four 10.0 gram samples of an ore: 1.254 grams, 1.2495 grams, 1.250 grams, and 1.2525 grams. Using the average of the four values given, what is the percentage of iron in the ore?
Ans. 12.5%
12. What is the value of 0.003 455 when rounded off to three significant figures? When rounded off to two significant figures? To one significant figure?

13. The Short Form for Division. Because skill in computation is a necessary part of the engineer's technical equipment, it is important that his working habits be highly efficient. In well managed factories, an old but usable machine is replaced as soon as a more efficient one becomes available. For similar reasons, it seems clear that an engineer must not cling to obsolete and wasteful mental tools after new and better ones have been devised.

In dividing one approximate number by another, the ordinary procedure is unnecessarily laborious. Let us examine in detail the process of dividing 2754 by 4388. It is convenient to put the divisor to the right of the dividend, and the quotient below the divisor. Carrying out the operation in the usual way, the work looks like this:

$$\begin{array}{r}
 27540\ 000 \quad |4388 \\
 26328 \qquad \quad 62762 \\
 \hline
 1212 \left\{ \begin{array}{l} 0 \\ 6 \end{array} \right. \\
 877 \left\{ \begin{array}{l} 6 \\ 40 \end{array} \right. \\
 \hline
 334 \left\{ \begin{array}{l} 40 \\ 16 \end{array} \right. \\
 307 \left\{ \begin{array}{l} 16 \\ 240 \end{array} \right. \\
 \hline
 27 \left\{ \begin{array}{l} 240 \\ 328 \end{array} \right. \\
 26 \left\{ \begin{array}{l} 328 \\ 912 \end{array} \right. \\
 \hline
 \end{array}$$

Rounding off the result according to rule, the answer is 0.6276.

Recalling the rule for addition and subtraction, we notice that, in the intermediate steps, only one column of doubtful figures should be retained. Thus, according to the rule, all figures to the right of the vertical wavy line should be rejected. The successive steps in this shortened form of division are as follows:

First Step.

$$\begin{array}{r} 2754 \quad |4388 \\ 26328 \quad 6 \\ \hline 1212 \end{array}$$

Second Step.

$$\begin{array}{r} 2754 \quad |4388 \\ 26328 \quad 62 \\ \hline 1212 \\ 878 \\ \hline 334 \end{array}$$

Third Step.

$$\begin{array}{r} 2754 \quad |4388 \\ 26328 \quad 627 \\ \hline 1212 \\ 878 \\ \hline 334 \\ 308 \\ \hline 26 \end{array}$$

Fourth Step.

$$\begin{array}{r} 2754 \quad |4388 \\ 26328 \quad 6276 \\ \hline 1212 \\ 878 \\ \hline 334 \\ 308 \\ \hline 26 \\ 24 \\ \hline \end{array}$$

It is evident that much of the drudgery of long division is eliminated by a rational application of the rules for computation. A further economy of effort may be effected. Referring again to the first step, we observe that

$$\frac{2754}{4388} = 0.6 + \frac{121.2}{4390}$$

The fraction in the right-hand member is evaluated as 0.0276 by slide rule

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(see below). Thus, using the rules for computation, and employing the slide rule to compute the last three significant figures, the complete operation of division may be performed as follows:

$$\begin{array}{r} 2754 \quad \overline{)4388} \\ 26328 \quad 6276 \\ \hline 1212 \end{array}$$

Suppose that the value of $\frac{37,728}{12,517}$ is required to five significant figures. Using the short form, the first two digits are calculated by longhand, and the last three by slide rule. The computation is given below by the old method and by the new:

$$\begin{array}{r} \textit{Short Form} \\ 37728 \quad \overline{)12517} \\ 37551 \quad 3.0141 \\ \hline 177 \end{array}$$

$$\begin{array}{r} \textit{Ordinary Form} \\ 377280000 \quad \overline{)12517} \\ 37551 \quad 3.0141 \\ \hline 17700 \\ 12517 \\ \hline 51830 \\ 50068 \\ \hline 17620 \\ 12517 \\ \hline 5103 \end{array}$$

14. The Short Form for Multiplication. The process of multiplying approximate numbers may be abbreviated in similar fashion. Carrying out the operation in the usual way, the result looks like this:

$$\begin{array}{r} 5.919 \\ 8.422 \\ \hline 11838 \\ 11838 \\ 2 \ 3676 \\ 47 \ 352 \\ \hline 49.849818 \end{array}$$

Rounding off according to rule, the result is 49.85. Instead of starting with the right-hand digit of the multiplier, it is better to begin with the digit on the left, and go *from left to right*, just as in ordinary algebra. The work

will then be as follows:

$$\begin{array}{r}
 5.919 \\
 8.422 \\
 \hline
 47\ 352 \\
 2\ 3676 \\
 11838 \\
 11838 \\
 \hline
 49.849818
 \end{array}$$

The last three places of decimals are meaningless. Accordingly, we may drop one significant figure from the multiplicand at each step. The successive steps are as follows:

First Step.

$$\begin{array}{r}
 5.919 \\
 8.422 \\
 \hline
 47\ 352
 \end{array}$$

Second Step.

$$\begin{array}{r}
 5.919 \\
 8.422 \\
 \hline
 47\ 352 \\
 2\ 368
 \end{array}$$

Third Step.

$$\begin{array}{r}
 5.919 \\
 8.422 \\
 \hline
 47\ 352 \\
 2\ 368 \\
 118
 \end{array}$$

Fourth Step.

$$\begin{array}{r}
 5.919 \\
 8.422 \\
 \hline
 47\ 352 \\
 2\ 368 \\
 118 \\
 12 \\
 \hline
 49.850
 \end{array}$$

The work may be further abbreviated by using the slide rule. Referring to the first step in the example under discussion, we observe that after multiplying 5.919 by 8, we must multiply 5.919 by 0.422 and add the result to the first one. By slide rule,

$$(5.92)(0.422) = 2.50$$

Hence the work in shortest form is as follows:

$$\begin{array}{r}
 5.919 \\
 8.422 \\
 \hline
 47\ 35\cancel{2} \\
 2\ 50 \\
 \hline
 49.85
 \end{array}$$

The advantages of the short form are even more apparent in multiplication of five figure numbers, as is shown by the following example:

<i>Short Form</i>	<i>Ordinary Form</i>
0.73752	0.73752
<u>44.917</u>	<u>44.917</u>
29 5008	516264
2 9501	73752
<u>676</u>	663768
33.1269	2 95008
	<u>29 5008</u>
	33.12718584
Rounded off: 33.127	Rounded off: 33.127

It is well to form the habit of estimating (by rough mental arithmetic) the answer to a problem *before* the computations are actually carried out. The position of the decimal point is in most cases easily determined in this way. In difficult cases, the position of the decimal point may be established with certainty by the method given in Chapter 9.

15. The Slide Rule. The slide rule is recognized as an indispensable aid in solving technical problems. So many sizes and kinds are now in use that detailed explanations are best obtained from the manuals of instruction provided by the manufacturer. The 10-inch Mannheim type is standard, and it is understood that this instrument or its equivalent is the slide rule referred to in this book.

The slide rule is especially well adapted to the solution of problems in ratio and proportion; and any problem that can be expressed as a proportion is readily solved by slide rule. In discussing the short form of division, we wished to obtain the value of $\frac{121.2}{4390}$. This problem in division can be expressed in proportion form thus:

$$\frac{121.2}{4390} = \frac{x}{1}$$

This proportion can be set up directly on the C and D scales; but it is best to form the habit of setting up every problem in division in such a way that the answer comes on the stock. Accordingly, the slide rule diagram looks like this:

$$\begin{array}{c|c|c} \text{C} & 4390 & \text{I} \\ \hline \text{D} & 1212 & x \end{array}$$

The hairline is set to 1212 on the D scale. The slide is then moved until 439 is under the hairline. The answer (0.0276) is then found on the D scale, opposite the index (1) on the C scale.

In our discussion of the short form of multiplication, it was desired to obtain the value of (5.92) (0.422). This problem in multiplication can be expressed in the form of a proportion thus:

$$\frac{1}{5.92} = \frac{0.422}{x}$$

The problem can now be set up directly on the C and D scales, in such a way that the answer is found on the stock. The slide rule diagram looks like this:

$$\begin{array}{c|c|c} \text{C} & \text{I} & 422 \\ \hline \text{D} & 592 & x \end{array}$$

The (right) index on the C scale is set to 592 on the D scale. The indicator is then moved until the hairline lies above 422 on the C scale. The answer (2.50) is then found under the hairline on the D scale.

The 10-inch slide rule is considered satisfactory for calculations to three significant figures, or to 1 part in 1000. The great majority of engineering calculations require no greater precision than this.

16. Radian Measure. The development of modern technology has brought into prominence a unit of angular measure, the radian, which at first appears awkward; but it turns out that many formulas take on their simplest form when angles are expressed in radians.

$$2\pi \text{ radians} = 1 \text{ revolution} \quad 2-1$$

$$\pi \text{ radians} = 180 \text{ degrees} \quad 2-2$$

$$1 \text{ radian} = 57.3 \text{ degrees, approximately} \quad 2-3$$

17. Arc and Central Angle. It is known from geometry that the length of arc of any circle is proportional to the central angle subtended by the

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arc. We have therefore (Figure 12):

$$\frac{\text{length of arc}}{\text{circumference of circle}} = \frac{\text{central angle}}{2\pi}$$

if the central angle is measured in radians. Thus

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$$

or

$$s = r\theta$$

2-4

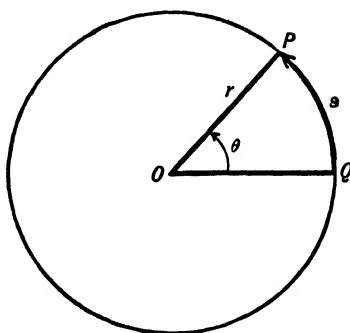


FIG. 12

18. Rim Speed and Angular Velocity. Suppose that the point P in Figure 12 is moving along the circumference of the circle with a speed v , called the peripheral or rim speed. If the speed is constant, we have

$$s = vt$$

where t is the time. Let the angular velocity of the radius OP be ω , so that

$$\theta = \omega t$$

Then

$$vt = s = r\theta = r\omega t$$

\therefore

$$v = r\omega$$

2-5

That is, the rim speed equals the angular velocity times the radius, provided that the angular velocity is expressed in radians turned through per unit time.

Exercises

1. From an observatory on the earth, the moon subtends an angle of about 31 minutes of arc. If the distance to the moon is 240,000 miles, what is its diameter? (Suggestion: Treat the moon's diameter as an arc of a circle.)
2. Taking the radius of the earth as 4000 miles (good to two significant figures), how large an angle would the earth subtend for an observer on the moon? *Ans.* $1^{\circ}55'$
3. If the diameter of an auto wheel is 44 inches, what is its angular velocity (in radians per second) when the car is moving at a speed of 88 ft. per second?
4. If the diameter of an iron flywheel is 11 ft., what angular velocity (in revolutions per minute) corresponds to a rim speed of 100 ft. per second (the limiting speed for safe operation)?
5. A wheel is turning with an angular velocity of 789 degrees per second. How fast is it traveling in radians per minute? *Ans.* 826
6. Find the number of degrees in 2.40 radians.
7. If a flywheel 2 feet 6 inches in diameter is turning with a constant angular speed of 22 radians per second, how many minutes will it take a point on the rim of the flywheel to travel a mile? *Ans.* 3.2
8. Find the number of radians in $62^{\circ}40'$.
9. Multiply the numbers 5.387 and 1.466, using the short form. Check by division.
10. Divide 34.725 by 6.3421, using the short form. Check by multiplication.

CHAPTER 3

SOLUTION OF RIGHT TRIANGLES

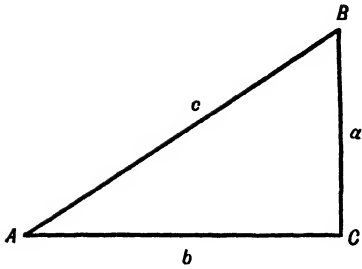


FIG. 13

19. Notation. The Two Principal Cases. It is convenient to use capital letters for the angles of a triangle, and lower case letters for the opposite sides, as in Figure 13. The letter *C* will denote the right angle. It is known from geometry that a right triangle is completely determined when any two of the five parts *a*, *b*, *c*, *A*, *B* are known, except when the two known parts are the two acute angles (because they are not independent).

When any two independent parts are known, the other three may be obtained by means of the formulas

$$A + B = 90^\circ \quad 3-1$$

$$a^2 + b^2 = c^2 \quad 3-2$$

$$a = c \sin A \quad 3-3$$

$$b = c \cos A \quad 3-4$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{c \sin A}{c \cos A} = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}} \quad 3-5$$

The last three of these follow directly from the defining equations 1-1, 1-2, and 1-3.

Of the various possible cases that can occur, two are especially important in practical work. When a problem recurs frequently, the engineer seeks that solution which entails the least effort. For these two cases there is required a method of solution that is efficient, and well adapted to solution by means of tables of natural trigonometric functions, to solution by slide rule, and, later, to solution by logarithms.

Case I. *Hypotenuse and one acute angle given.* The solution is obtained by equations 3-1, 3-3, and 3-4. Equation 3-5 affords a convenient check. Let $c = 234.1$ and $A = 32^\circ 17'$ (see Figure 14 on page 28).

<i>Solution</i>	<i>Computation</i>
$a = c \sin A$	2341
	53411
$= 234.1(0.53411)$	11705
	798
$= 125.0 \text{ Ans.}$	12503
$b = c \cos A$	2341
	84542
$= 234.1(0.84542)$	18728
	1063
$= 197.9 \text{ Ans.}$	19791
$B = 90^\circ - A$	89°60'
	32°17'
$= 57^\circ 43' \text{ Ans.}$	57°43'

Check

$\tan A = \frac{a}{b}$	12503	19791
	11874	6318
	629	
0.63177	125.03	
	197.91	
	0.6318	

Solution of Right Triangles

Case II. Two legs given. The solution is obtained by equations 3-5, 3-3, and 3-1. Equation 3-4 affords a convenient check. Let $a = 23.71$ and $b = 35.43$ (see Figure 15 on page 29).

<i>Solution</i>	<i>Computation</i>
$\tan A = \frac{a}{b}$	$\begin{array}{r} 2371 \quad \quad 3543 \\ \hline 21258 \quad 6692 \\ \hline 2452 \end{array}$
$= \frac{23.71}{35.43}$	
$= 0.6692$	
$A = 33^\circ 47' \quad \text{Ans.}$	
$c = \frac{a}{\sin A}$	$\begin{array}{r} 2371 \quad \quad 5561 \\ \hline 22244 \quad 4263 \\ \hline 1466 \end{array}$
$= \frac{23.71}{0.5561}$	
$= 42.63 \quad \text{Ans.}$	
$B = 90^\circ - A$	$\begin{array}{r} 89^\circ 60' \\ 33^\circ 47' \\ \hline 56^\circ 13' \end{array}$
$= 56^\circ 13' \quad \text{Ans.}$	
<i>Check</i>	
$b \stackrel{?}{=} c \cos A$	$\begin{array}{r} 4263 \\ 8312 \\ \hline 34104 \\ 133 \\ \hline 3543 \end{array}$
$35.43 \quad \quad 42.63(0.8312)$	
35.43	

In solving triangles, it is well to remember that a good sketch promotes clear thinking. The triangle should be drawn approximately to scale.

It is good practice to show all computations, other than those performed mentally, together with the rest of the solution. The arrangement of work on the page should be planned in such a way that another person can see at a glance (a) the results, (b) the method and formulas used, and (c) each step in the computation. It may be observed that the Pythagorean relation 3-2 is not used, because it is not well adapted to computation. Notice also that, in computations to four significant figures, it is preferable to use five place tables, in order to avoid interpolation.

20. Checking. The habit of checking involves more than the acquisition of certain techniques. It is part of the basic philosophy of engineers and scientists.

Psychologists say that our thoughts are shaped by our desires. It is a fact of everyday experience that many a problem is wrongly solved, and yet the check appears to confirm the wrong answer. *When we think we know the answer, our solutions and checks subconsciously tend to confirm this answer, whether right or wrong.*

Imagine yourself checking the work of someone else, someone disliked, so that it would give you pleasure to expose his error. It would be safe to wager that, under such conditions, if there is a mistake, you will find it. It is just this attitude which we must cultivate toward our own work: we must learn to expect mistakes, and to find pleasure in detecting them. Legend has it that an elephant tests every timber and every step when crossing a bridge. So the engineer must cultivate the habit of intellectual skepticism with regard to his professional problems; he must carefully test every step in the bridge of reasoning connecting a problem with its solution.

The following criteria may serve as a guide in selecting one from a variety of checks that may be available. The relative weights assigned to the several criteria will naturally vary with the circumstances.

(a) *Completeness.* A check is complete if all answers are checked against original data. The check suggested for the Case I triangle is incomplete, since angle B is not checked. The same remark applies to the check for the Case II triangle. (A complete check can easily be devised; but the second angle is seldom required in practical work.) Logarithmic checks are often incomplete, when the logarithms of the answers are checked against the logarithms of the given data. Note that errors made in transcribing data, as from the page of a textbook to the student's paper, will not be caught by checking answers against the transcribed data.

(b) *Independence.* The ideal check, with respect to completeness as well as independence, is a separate solution to the problem, made by another individual using an entirely different method. In checking one's own work, it is desirable to employ an independent formula or method for two reasons; to guard against theoretical errors, or errors in principle; and to guard against the tendency to repeat errors when repeating an operation. Every operation of subtraction should be checked by addition. If a column of figures is added from the top down, it should be checked by

adding from the bottom up. In many kinds of engineering work it is good practice to solve a problem analytically, and check by a graphical solution, or vice versa.

(c) *Length*. Usually a choice must be made from several available methods of checking. As between two otherwise equally satisfactory methods, the less cumbersome is of course to be preferred. This criterion is closely related to the next.

(d) *Precision*. Usually, the greater the precision sought, the more the time and effort required. **A rough check which ought always to be applied is examining the results to see if they are reasonable.** Thus, a glance at Figure 14 shows that side a should be less than side b , and both should of course be less than the hypotenuse. In the same way, it is apparent from Figure 15 that angle A is less than 45° , and that c is greater than either a or b . A mistake in setting the decimal point, that *bête noire* of the engineering student, can usually be caught by checking for reasonableness. A check by slide rule is more precise, but takes more time. The slide rule is often useful in step-by-step checking, even though a more precise check may be necessary for the problem as a whole.

(e) *Reliability*. Obviously the purpose of any check is to discover errors; and the ideal check would be one certain to reveal any error in the solution. Certainty in this uncertain world is difficult of attainment; but simple checks are more reliable than complicated ones; and well tried formulas and methods are to be preferred over unusual or special ones. It is possible to test a proposed check for reliability by studying the effects of deliberate errors; but the variety of possible errors and combinations of errors is so great that, in any but the simplest problems, a conclusive demonstration of reliability is not practicable.

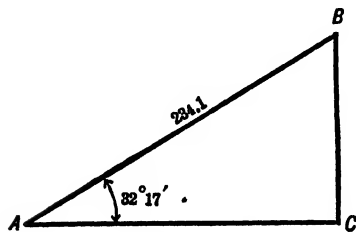


FIG. 14

21. Solution of Right Triangles by Slide Rule. When the Mannheim type slide rule is used for solving triangles, it is best to take out the slide and turn it over, so that the S scale (scale of sines) lies opposite the A scale, to which it corresponds. The T scale (scale of tangents) is then opposite the D scale, to which it corresponds.

Case I. Hypotenuse and one acute angle given. Let $c = 234$ and $A = 32^\circ 15'$ (see Figure 14). From equation 3-1 we find $B = 57^\circ 45'$. Equations 3-3 and 3-4 can be expressed

in the form of a proportion with the aid of equation 1-4:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{1}$$

The slide rule diagram is as follows:

$$\frac{A}{S} \left| \frac{a = ?}{32^\circ 15'} \right| \frac{b = ?}{57^\circ 45'} \left| \frac{234}{I} \right|$$

Set the index on the S scale to 234 on the A scale. Opposite $32^\circ 15'$ on the S scale read $a = 125$. Opposite $57^\circ 45'$ read $b = 198$. This checks our previous solution to three significant figures

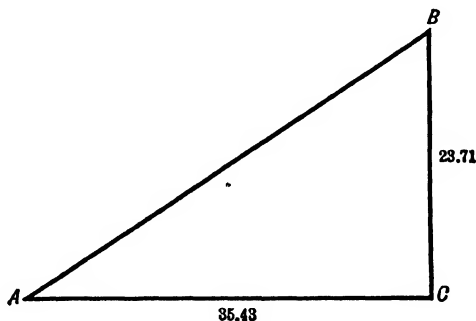


FIG. 15

Case II. *Two legs given.* Let $a = 23.7$ and $b = 35.4$ (see Figure 15). Since the scale of tangents reads directly only to 45° , it is best to find the smaller of the two acute angles first. Equation 3-5 can be expressed in the form of a proportion as follows:

$$\frac{\tan A}{a} = \frac{1}{b}$$

The slide rule diagram is

$$\frac{T}{D} \left| \frac{A = ?}{237} \right| \frac{I}{354}$$

Set the index on the scale of tangents opposite 354 on the D scale. Opposite 237 on the D scale we find $33^\circ 50'$.

From equation 3-3 we now have the slide rule diagram

$$\frac{A}{S} \left| \frac{237}{33^\circ 50'} \right| \frac{c = ?}{I}$$

Set the indicator to 237 on the A scale. Move the slide until $33^{\circ}50'$ on the S scale lies under the hairline. Opposite the index on the S scale read $c = 42.6$. This checks our previous solution to three significant figures.

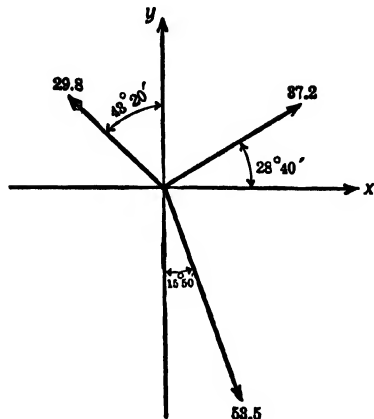


FIG. 16

22. Finding the Resultant of Several Forces. To illustrate how the foregoing methods are employed in solving technical problems, suppose that three forces act on a small object. The magnitudes and directions of the forces are represented by line vectors in Figure 16. It is required to find the resultant (or vector sum) of the three forces. The analytical solution of this problem is to find the components of each force in the x -direction and in the y -direction. The algebraic sum of the x -components gives the resultant force in the x -direction; similarly, the sum of the y -components gives the resultant force in the y -direction. In Figure 17(a) the components of each

in the x -direction; similarly, the sum of the y -components gives the resultant force in the y -direction. In Figure 17(a) the components of each

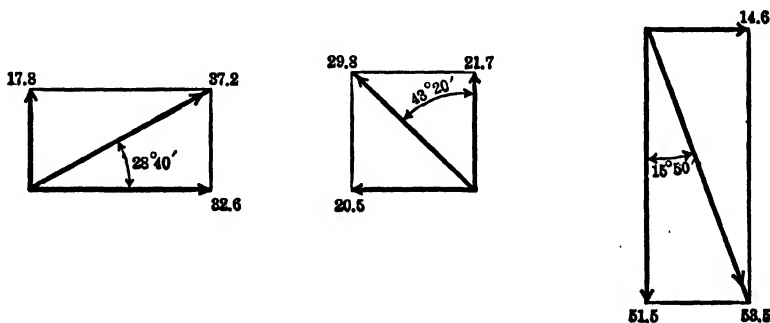


FIG. 17(a)

force are shown; they are calculated to three significant figures by the method of the previous section (Case I). The resultant force in the x -direction is

$$\begin{aligned} R_x &= 32.6 - 20.5 + 14.6 \\ &= +26.7 \text{ lb.} \end{aligned}$$

The resultant in the y -direction is

$$\begin{aligned} R_y &= 17.8 + 21.7 - 51.5 \\ &= -12.0 \text{ lb.} \end{aligned}$$

The resultant force has the magnitude and direction shown in Figure 17(b). The calculation is made by slide rule by the method of the previous section (Case II).

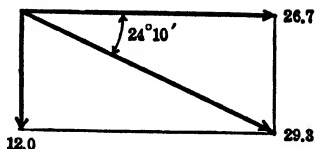


FIG. 17(b)

23. Angle of Elevation. For an observer at a point O , the angle of elevation of a point P is the angle made by the line OP with the horizontal. It is understood that P is higher than O .

24. Angle of Depression. Defined just as above, except that P is understood to be lower than O (Figure 18).

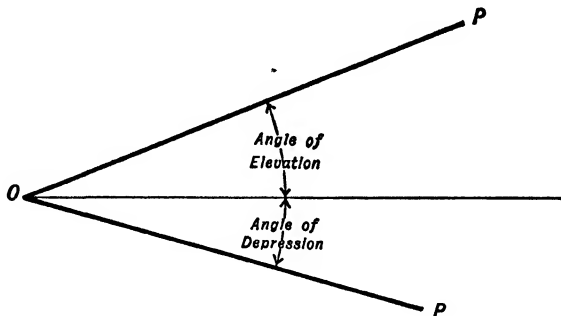


FIG. 18

Exercises

Solve and check the following triangles. Angle C is a right angle.

1. $c = 0.0133$

$A = 28^{\circ}20'$

3. $a = 4.87$

$b = 5.62$

5. $a = 712$

$A = 37^{\circ}40'$

7. $a = 396$

$c = 513$

2. $c = 4822$

$B = 51^{\circ}23'$

4. $a = 28.48$

$b = 71.99$

6. $b = 0.4115$

$A = 44^{\circ}14'$

8. $b = 4.122$

$c = 4.926$

Ans. $a = 3009$

$b = 3768$

Ans. $A = 21^{\circ}35'$

$c = 77.42$

9. A vector has a horizontal component $v_x = 330$, and a vertical component $v_y = -270$. Find the magnitude of the vector, and the angle which it makes with the horizontal.

Ans. $430, 39^{\circ}15'$

10. A vector has a horizontal component $v_x = -240$, and a vertical component $v_y = 450$. Find the magnitude of the vector, and the angle which it makes with the horizontal.
11. The rectangular components of a force are 35 lb., acting at an angle of 15° with the positive x -axis, and 44 lb., at an angle of 105° with the x -axis. Find the magnitude and direction of the resulting force. *Ans.* 56 lb., $66^\circ 30'$
12. The rectangular components of a force are 2300 lb., acting at an angle of 150° with the positive x -axis, and 1800 lb., acting at an angle of 240° with the x -axis. Find the magnitude and direction of the resulting force.
13. An airplane must be held to a course due west, against a wind velocity of 60 miles per hour coming from the southwest. In what direction must the plane be headed, and what will be the ground speed, if the air speed is 240 miles per hour? (Suggestion: Find the components of the wind speed. What must the southerly component of the airplane's air speed be?) *Ans.* $S 79^\circ 50' W$, 190 m.p.h.
14. Find the length of the slender rod $ABCD$ in Figure 19. The curved part is a semi-circle.

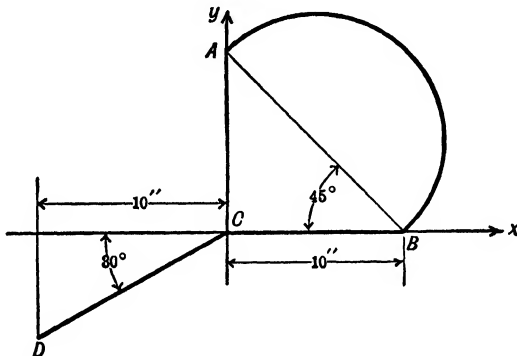


FIG. 19

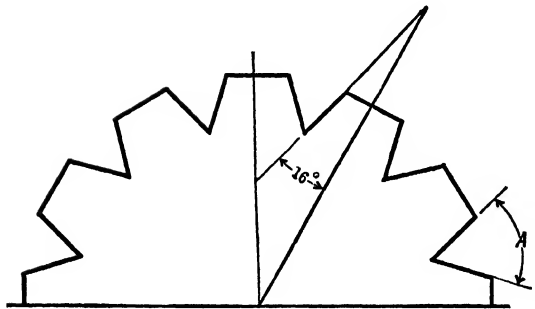


FIG. 20

15. An airplane flying at a constant altitude of 2100 feet passed directly over a point A on the ground. At a certain instant later the angle of depression of A was $38^\circ 15'$ and 3 seconds afterward it was 30° . How far did the airplane move during the 3 seconds?

Ans. 980 feet

16. Determine angle A for the machine part shown in Figure 20.

17. A polygon with five equal sides, and equal angles, has a perimeter of 235 inches. Find its area, and the area of the circle circumscribed about it.
Ans. 3800 and 5030
18. A tower standing on level ground casts a shadow 216 feet long. At a distance of 462 feet from the base of the tower the angle of elevation of the top is $26^{\circ}15'$. Find the angle of elevation of the sun.

Miscellaneous Problems

1. The following numbers are given to five significant figures. Round off each of them to three significant figures.

0.317 53	3.8850
2.7183	3.8860
1,005,100	3.8750
1,005,000	0.068 349

2. Find the projection of a line segment 16 inches long upon a straight line, if the straight line passes through the line segment at a point 6 inches from one end, and at an angle of 31° .
3. The diameter of a circle is measured as 27 inches. What is its circumference? If the diameter were given as 27.0, what value should be given for the circumference? What if the diameter were 27.00?
4. The president of a large manufacturing company plans to put out a new product. He requests an estimate from the cost department and receives a report of \$5468.72 total cost per gross of the manufactured article, itemized as follows:

Raw materials	\$3368.72
Labor	1800.00
Overhead (including lubrication, fuel, janitor service, etc.)	300.00
	<hr/>
Total cost	\$5468.72

What criticism could be made of the above report, and how should it read when properly revised?

5. How long a brace is needed to reach from the top of a pole 62 feet high to the ground, if the brace is to make an angle of 58° with the ground?
6. If the driving wheels of a locomotive are 15 feet in circumference, how many revolutions must they make per minute so that the locomotive may attain a speed of 55 miles per hour?
7. The side of a square was estimated to be 3.5 inches. By actual measurement it was 3.6 inches. Find the area of the square as determined from both the estimated and the measured values for the side. What are the percentage errors in the estimated side and area? Do not give more significant figures in your answers than are justified.
8. What is the angular velocity of the second hand of a watch in degrees per second? In radians per second? How many radians does the minute hand pass through in 40 minutes?

9. A 16-pole synchronous motor is running at 450.0 revolutions per minute. Calculate, to four significant figures, the angular velocity in radians per second. Find the linear velocity of a strap on the armature, at a distance of 1.55 feet from the axis.
10. The wheels of a bicycle are 26.0 inches in diameter. It is so geared that two revolutions of the feet correspond to three revolutions of the wheels. How many revolutions of the feet per minute are necessary to ride with a speed of 18 miles per hour? Suppose that the tires are not well inflated; what is the percent of loss in distance if the compression in each tire is 0.50 inches?
11. Evaluate by slide rule:
- $$(a) \frac{(39.7)(2.46)(0.732)}{(0.564)(0.0311)}$$

$$(b) \sqrt{0.008} \quad 14$$

$$(c) 23.8^{\circ}$$

$$(d) \frac{(0.477)\sqrt{0.0355}}{8340}$$
12. Change $\frac{8\pi}{3}$ radians to degrees. How many radians are there in $101^{\circ}20'$?
13. A ladder 17.5 feet long just reaches the top of a wall when inclined at an angle of $55^{\circ}30'$ with the horizontal. How high is the wall?
14. The diameter of the earth's orbit about the sun (186,000,000 miles) subtends at the nearest fixed star an angle of approximately 1.53 seconds of arc. Find the distance to the star.
15. Find the area of an isosceles triangle in which the equal sides, 14 inches in length, include an angle of 120° .
16. A force of 130 lb. makes an angle of 32° with the x -axis. A force of 220 lb. makes an angle of 61° with the x -axis. Both forces act in a direction upward to the right. Find the magnitude and direction of the resultant of the two forces.
17. An airplane is in level flight in a direction $N 28^{\circ}40' E$, at a speed of 280 miles per hour. What is the component of the speed in a northerly direction? In an easterly direction?
18. From the top of a tower 256 feet high, the angles of depression of two points on the ground are $7^{\circ}15'$ and $12^{\circ}50'$. If they are on the same side of the tower and in line with its base, how far apart are they?
19. Two forces, one of 410 lb. and the other of 320 lb., make an angle of $49^{\circ}40'$. Find the magnitude and the direction of their resultant.
20. As part of the inspection routine in the production of a certain machine part, a measuring wire is inserted as shown in Figure 21. The three sides should be tangent to the wire at one time. Determine the proper diameter of the measuring wire.
21. A man walks a horizontal distance of 2.0 miles to the northeast, followed by 1.5 miles directly east, and then 3.0 miles to the southeast. What are his distance and bearing by the compass from the starting point? Observe that each part of the journey can be represented by a vector; the vector sum is required.
22. A new photographic device makes use of light from two point sources, which, passing through a slotted screen, falls upon a photographic plate, in such a manner that the light from one source does not overlap with that from the other, but the entire plate is illuminated by either one source or the other, as shown in Figure 22. If each slot is 0.06 inches wide, how should they be spaced?

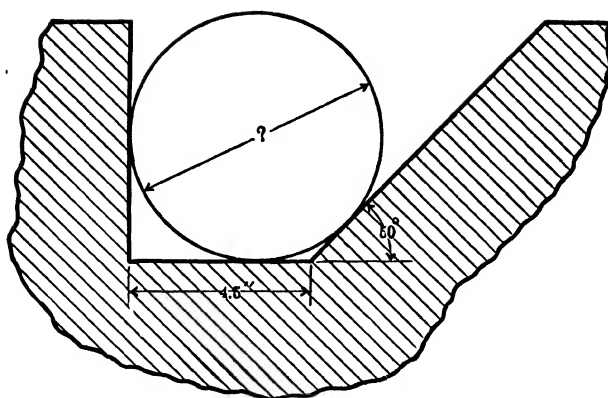


FIG. 21

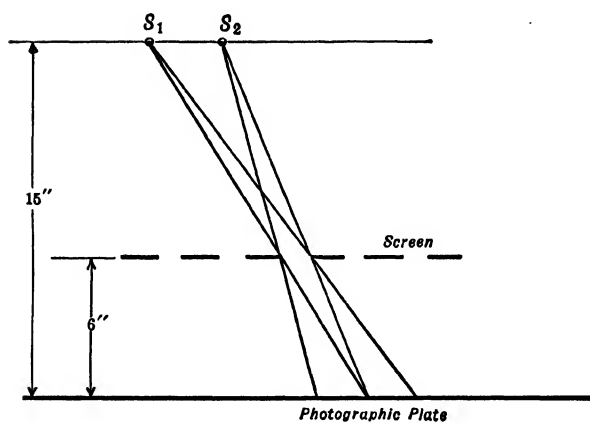


FIG. 22

23. Light from a point source passes through a wire mesh screen and falls on a photographic plate as shown in Figure 23. The wires are 0.01 inches in diameter. Calculate the width of the shadow cast by the wire *A*.

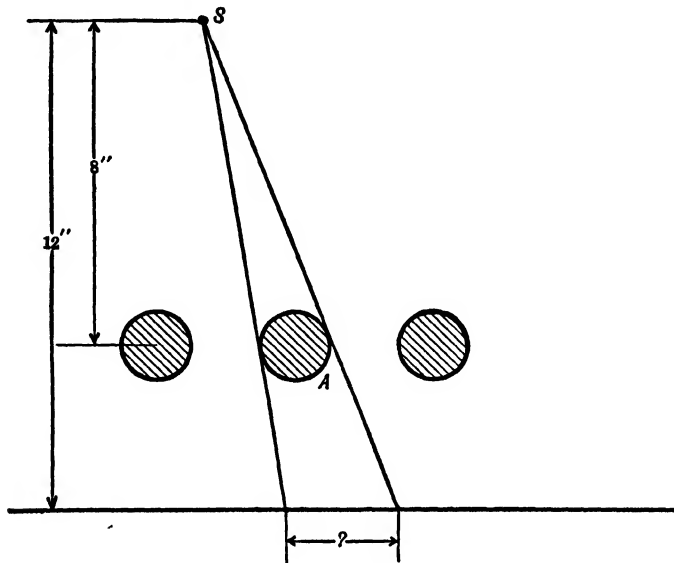


FIG. 23

CHAPTER 4

ROTATING VECTORS

25. Periodic Phenomena and Clocks. In many respects, the electric clock exemplifies to a high degree certain essential characteristics of all machines. In machines, and in nature, too, we observe that certain phenomena recur again and again in a regular way. In the broader sense any regularly recurring, or periodic, phenomenon may be used as a clock. For the purposes of the physicist, atoms furnish excellent clocks, because atomic behavior is periodic in most respects, the periods maintaining a high degree of constancy. Water waves, sound waves, and radio waves are other familiar examples of periodic phenomena. The rotation of the earth about its axis furnishes us with a master clock with which all other clocks are compared. As a typical example, representative of the whole class of periodic phenomena, the electric clock serves very well.

26. Geometrical Representation of the Sine and Cosine Functions. Imagine, then, an electric wall clock with a large red indicator which makes a complete revolution every minute. We may visualize the mathematical concept of a *rotating vector* in terms of the indicator, uniformly turning on the dial of the clock. If in particular the length of the indicator is taken to be one unit, we obtain the concept of a *rotating unit vector*. The *counterclockwise* direction of rotation is usually taken to be positive, and the clockwise direction is considered negative.

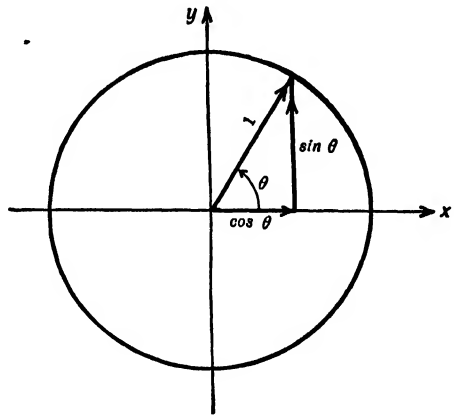


FIG. 24

In Figure 24 there is shown a rotating unit vector, with its horizontal and vertical components. It is customary to specify the position of the vector by the angle made with the positive x -direction; a positive angle is laid off counterclockwise, as shown. Negative angles are laid off in the opposite (clockwise) direction. An angle so represented is said to be laid off in standard position.

The horizontal projection of a rotating unit vector represents the cosine of an angle laid off in standard position. The vertical projection represents the sine of the angle. This follows directly from the definitions of Chapter 1. In developing the implications of those definitions we have as yet encountered no angles larger than 90° . Our present point of view, however, inevitably suggests that larger angles (and negative angles) may be encountered, and at the same time affords means for studying the trigonometric functions of such angles.

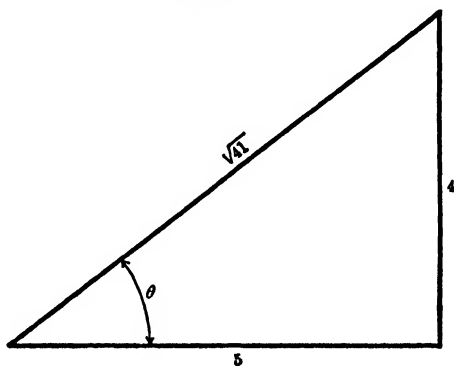


FIG. 25

27. The Pythagorean Identities. From Figure 24 it is immediately evident that

$$\sin^2 \theta + \cos^2 \theta = 1 \quad 4-1$$

for all values of θ . (Note that $\sin^2 \theta$ means the square of the sine of θ .) By simple algebraic operations the identity 4-1 may be expressed in such equivalent forms as

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta}$$

$$\cos \theta = \pm \sqrt{(1 + \sin \theta)(1 - \sin \theta)}$$

Let us define two new trigonometric functions, the secant and the cosecant, by the equations

$$\sec \theta = \frac{1}{\cos \theta} \quad 4-2$$

$$\csc \theta = \frac{1}{\sin \theta} \quad 4-3$$

If both members of the identity 4-1 be divided by $\cos^2 \theta$, the result is

$$\frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \frac{1}{\cos^2 \theta}$$

$$\therefore \tan^2 \theta + 1 = \sec^2 \theta \quad 4-4$$

If both members of 4-1 be divided by $\sin^2 \theta$, the result is

$$1 + \cot^2 \theta = \csc^2 \theta \quad 4-5$$

With the aid of the Pythagorean identities 4-1, 4-4, and 4-5, it is possible to express any of the six trigonometric functions in terms of any one of the remaining functions of the same angle.

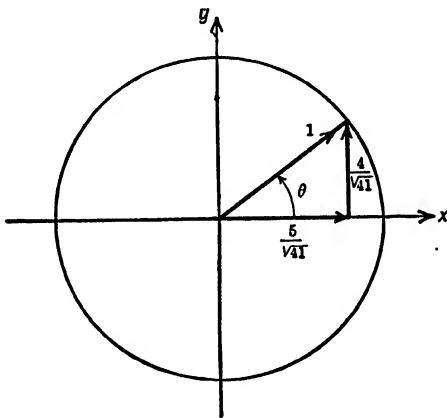


FIG. 26

Example. Let $\tan \theta = \frac{4}{5}$, with θ less than 90° ; it is required to find the values of the remaining trigonometric functions.

Recalling that, in a right triangle,

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

(equation 3-5), we may construct a right triangle, one of whose angles is θ , as in Figure 25. Next, we lay off a unit vector making the same angle, as in Figure 26. This exhibits $\sin \theta$ and $\cos \theta$ directly; the remaining functions are

calculated from their defining equations.

$$\sin \theta = \frac{4}{\sqrt{41}} \qquad \csc \theta = \frac{\sqrt{41}}{4}$$

$$\cos \theta = \frac{5}{\sqrt{41}} \qquad \sec \theta = \frac{\sqrt{41}}{5}$$

$$\tan \theta = \frac{4}{5} \qquad \cot \theta = \frac{5}{4}$$

Exercises

1. Find the values of the other five functions if $\tan A = \frac{5}{7}$ and A is an acute angle;

2. Find the values of the other five functions if $\cos B = \frac{2}{3}$ and B is an acute angle;

3. Express $\sin A$ in terms of $\tan A$.

$$\text{Ans. } \frac{\tan A}{\sqrt{1 + \tan^2 A}}$$

4. Express $\cos B$ in terms of $\tan B$.

5. If for an acute angle A ,

$$\sin A = \frac{2xy}{x^2 + y^2}$$

find a formula for $\tan A$ in terms of x and y .

$$\text{Ans. } \frac{2xy}{x^2 - y^2}$$

6. If for an acute angle B ,

$$\cos B = \frac{\sqrt{2xy}}{x + y}$$

find a formula for $\cot B$.

7. Simplify the expression

$$\left(\tan \phi + \frac{\cos \phi}{\sin \phi} \right) \cos \phi$$

$$\text{Ans. } \csc \phi$$

8. Simplify the expression

$$\frac{\tan x + \cot x}{\csc x}$$

9. Reduce

$$\frac{\sec^2 B - \sin B \sec B - 1}{\tan B - 1}$$

to a form containing only the tangent function.

28. Functions of 45° . Tables of the trigonometric functions are in general computed by methods beyond the scope of this book. There are some special cases, however, in which the values of the six trigonometric functions may be calculated by elementary methods.

The angle of the unit vector shown in Figure 27 is 45°. From the theorem of Pythagoras (equation 3-2) the equal legs are readily found to have the value $\frac{1}{\sqrt{2}}$. Hence

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1$$

$$\cot 45^\circ = \frac{1}{\tan 45^\circ} = 1$$

$$\sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2}$$

$$\csc 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2}$$

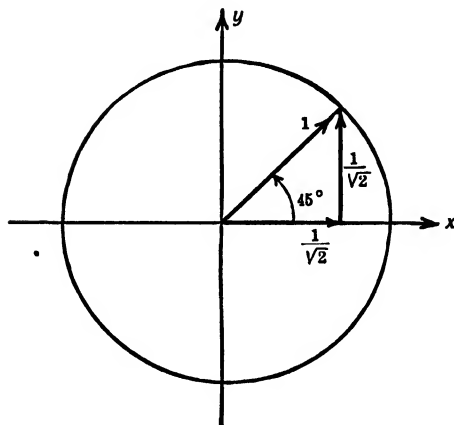


FIG. 27

29. Functions of 30° and 60°. It is known from elementary geometry that if the angles of a triangle are 30°, 60°, and 90°, the hypotenuse is double the shorter leg in length. The other leg is readily found by equa-

tion 3-2. From Figure 28

$$\sin 30^\circ = \frac{1}{2}$$

$$\csc 30^\circ = 2$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sec 30^\circ = \frac{2}{\sqrt{3}}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\cot 30^\circ = \sqrt{3}$$

From Figure 29, we see that

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\csc 60^\circ = \frac{2}{\sqrt{3}}$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\sec 60^\circ = 2$$

$$\tan 60^\circ = \sqrt{3}$$

$$\cot 60^\circ = \frac{1}{\sqrt{3}}$$

30. Behavior of the Cosine Function. Let us observe how the horizontal component changes as the unit vector rotates. In Figure 30 the unit vector is shown in four cardinal positions. For 0° and 180° , and for any *even* multiple of 90° , the horizontal component is identical with the rotating vector itself. For 90° and 270° , and for any *odd* multiple of 90° , the horizontal component vanishes.

As the unit vector turns from its initial position to the 90° position, the cosine decreases, slowly at first, then faster and faster, to vanish at 90° . As the angle increases from 90° to 180° , the horizontal component is directed to the left; to distinguish this from the case where the component is directed to the right, we make the convention that *vector quantities directed to the left are negative*. Hence, in quadrant II, the cosine decreases (algebraically) from 0 to -1 . In quadrant III, the cosine goes from -1 back to 0. In quadrant IV, the cosine once more becomes positive, and increases from 0 to $+1$. At 360° the initial position is attained, the cycle is completed, and another identical cycle begins.

The motion represented by the horizontal component of the rotating vector is called **simple harmonic**. It is exhibited in nature by a weight bobbing on the end of a spring; by the vibrations of an atom in a crystal lattice; by the motion of our ear drums which causes the sensation that we recognize as sound; by the to-and-fro surging of electrons in an alternating current circuit; and by many other periodic phenomena. Although

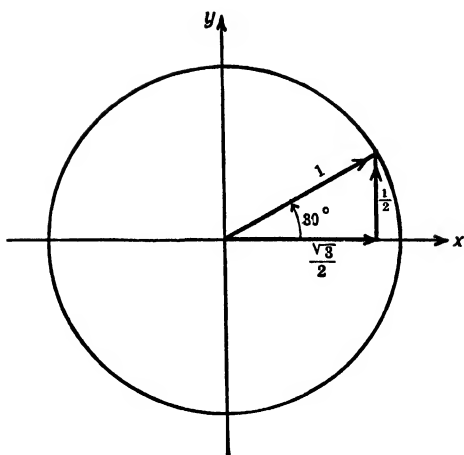


FIG. 28

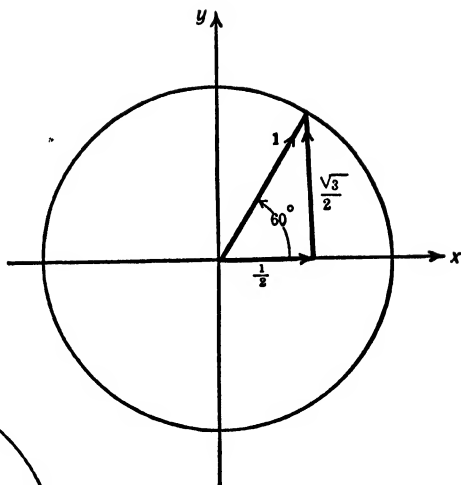


FIG. 29

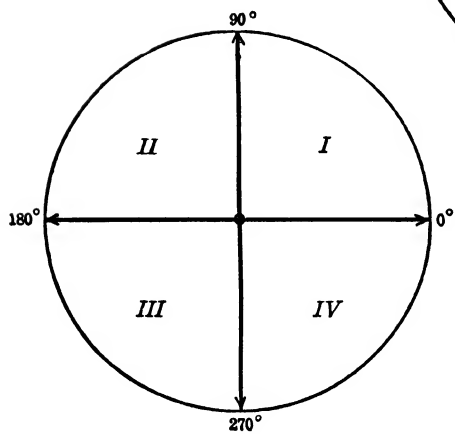


FIG. 30

the cosine function was originally invented for the purpose of solving triangles, its chief importance in applied mathematics, in modern times, is owing to the fact that it represents simple harmonic motion analytically. But the further development of analytic trigonometry will be postponed until certain algebraic ideas have been discussed.

Exercises

- Trace the behavior of the sine function as the angle increases steadily from 0° to 360° .
- Fill in the following table giving the signs of the trigonometric functions in the various quadrants:

	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
sine				
cosine	+	-		
tangent				

- Find the numerical value of $\sin^2 60^\circ + \cos^2 45^\circ$.
- Find the numerical value of $\tan 45^\circ + 3 \cos 60^\circ$.
- A force of 325 lb. acts upward to the left, at an angle of 30° with the horizontal. Find the horizontal and vertical components.
- A force of 0.54 tons acts downward to the right, making an angle of 60° with the horizontal. Find the horizontal and vertical components.
- Simplify the expression

$$\sin^2 x + \frac{\cos x + \tan x}{\sec x}$$

Ans. $1 + \sin x$

- Simplify the expression

$$1 + \sin^2 A \sec A - \frac{\tan A}{\sin A}$$

and check by substituting $A = 30^\circ$ in the given expression, and in your answer.

CHAPTER 5

THE LANGUAGE OF ALGEBRA

31. The Idea of Function. In our discussion of the trigonometric functions thus far, an intuitive understanding of the idea of function has been taken for granted. We now pause to examine the concept of function itself, in order that subsequent discussion may be both simpler and more precise.

The idea of function, like the concept of number, is implicit in almost all mathematical thought. We are usually concerned, not with a single quantity, but with two or more *related* quantities. A mathematical relationship between two quantities, say y and x , may be expressed in various ways: (a) by means of a *table of corresponding values* for y and x ; or (b) by a *graph* of some kind, in which y is plotted against x ; or (c) by a *formula* or equation. By definition, *a quantity y is a function of another quantity x if, when x is given, y is determined.*

32. The f -Notation. It is often convenient to indicate that a functional relationship exists by writing

$$y = f(x)$$

which is merely an abbreviation for the statement that y is some function of x . This notation is especially useful when we wish to speak, not about a particular function, but about a class of functions. Thus, we may wish to investigate certain properties common to all of the trigonometric functions; then $f(x)$ would represent any one of the six. Or we might wish to discuss the class of algebraic polynomials; then $f(x)$ would be understood to stand for any expression such as $ax^2 + bx + c$ or $19x^4 - 33x^2 + 5x + 54$.

The development of mathematics is characterized by the use of symbols representing concepts of ever-increasing generality. A great advance was made when men began to talk about *numbers*, such as 6, or 17, or 89, instead of *things* such as six cows, or seventeen knives, or eighty-nine houses. Another great advance was made when a letter, such as x , was caused to stand for *any number*. This marked the beginning of algebra. At first only a few men of exceptional ability were able to grasp such abstractions; now it is expected of every boy in high school that he be able to use and understand simple algebraic formulas. The idea of using

a symbol to stand for an *entire class of functions* is a generalization at a still higher level, which, after a little practice, will be found most useful.

It must be clearly understood that $f(x)$ is a single symbol; it does *not* mean that a quantity f is to be multiplied by a quantity x .

The quantity x is called the *argument* of the function, or the *independent variable*. The functional notation affords a convenient means of representing the value of the function when the argument takes on some particular value. If $f(x)$ represents any function, then $f(a)$ represents the value of the function when x is replaced by a .

Example 1. If $f(x) = \tan x$, what is $f\left(\frac{\pi}{4}\right)$?

Here x takes on the value $\frac{\pi}{4}$ radians, or 45° .

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= \tan \frac{\pi}{4} \\ &= 1 \end{aligned}$$

Example 2. If $f(x) = \sin x$, what is the simplest form of $f\left(\frac{\pi}{2} - x\right)$?

We must replace x by $\left(\frac{\pi}{2} - x\right)$.

$$\begin{aligned} f\left(\frac{\pi}{2} - x\right) &= \sin\left(\frac{\pi}{2} - x\right) \\ &= \cos x \end{aligned}$$

Example 3. If $F(x) = \frac{a+x}{a-x}$, what is the value of $F\left(\frac{a+x}{a-x}\right)$?

The meaning of $F\left(\frac{a+x}{a-x}\right)$ must be clearly grasped; we are directed to substitute $\left(\frac{a+x}{a-x}\right)$ for (x) in the formula for $F(x)$. Hence

$$\begin{aligned} F\left(\frac{a+x}{a-x}\right) &= \frac{a + \frac{a+x}{a-x}}{a - \frac{a+x}{a-x}} \\ &= \frac{a^2 - ax + a + x}{a^2 - ax - a - x} \end{aligned}$$

33. Algebra Is a Language. In a textbook on algebra, the formal operations of algebra are necessarily emphasized. The purpose and meaning of the operations, to, let us say, the engineer, cannot always be made plain. Thus the foregoing discussion of functional notation is intended to show the *kind* of operations that are permitted with the symbol $f(x)$. The purpose and meaning of the notation can be made clear only by the use to which it is put, in this course, and in succeeding ones.

One of the most eminent of living mathematicians, Bertrand Russell, confesses that when he began the study of algebra, there was one thing that especially bothered him: he could never find out what x was. He adds that his impression at the time was that his teacher knew, but wouldn't tell! The rest of us may take some comfort in the thought that even genius finds it easier to learn the manipulations of algebra than to get an understanding of what algebra means.

Learning to use algebra is very much like learning a foreign language. *Algebra is the language of quantity.* In principle, any idea that can be expressed in algebraic symbols can be expressed in plain English; and every possible operation of algebra can be explained in words, using the ordinary canons of logic. An equation, from this point of view, is merely an algebraic sentence.

All of the operations and symbols of algebra have been especially designed to promote simple, logical, thought; hence the scientist and the engineer tend to do as much thinking as they possibly can in the language of algebra.

The student ought to form the habit of translating all new algebraic ideas and principles into plain English. It is a valuable aid in understanding just what is meant; and it cannot fail to impress one with the simplicity, precision, and elegance of algebraic language in expressing quantitative relations.

As an illustration, let us take the formula $s = r\theta$ (equation 2-4). The meaning of this equation may be put in words as follows: *The length of an arc of a circle is proportional to the central angle subtended by it. It is also proportional to the radius of the circle.* An alternative translation is this: *The length of an arc of a circle, in any units of length, is equal to the product of the radius, expressed in the same units, and the central angle subtended by the arc, when the angle is measured in radians.*

Should we wish to emphasize one aspect of this relationship, we might write

$$s = f(\theta)$$

The translation is: *The arc of a circle is a function of the central angle sub-*

tended by it. If the angle be taken to be one radian, the algebraic expression for the result is

$$s = f(1) = r$$

The translation of this equation is often used as the definition of a radian: *The arc intercepted by a central angle of one radian is equal in length to the radius.*

34. Operations Upon Fractions. The student already has some knowledge of the elementary operations, which may be called the basic grammar of algebra. Some of the exercises in this chapter (and in subsequent ones) are intended to provide an opportunity for reviewing, and attaining a more complete mastery of, elementary algebra.

In order to benefit fully from these exercises, it is essential that the student form the habit of asking himself, at every stage, *How can this step be justified?* Unless one sees the point of the exercise or problem, much of the value may be lost. Mere unthinking repetition of the mechanical operations of algebra never taught anyone anything.

Example 1. Simplify the expression

$$\frac{\frac{1}{7} + \frac{1}{6}}{1 + (\frac{1}{7})(-\frac{1}{6})}$$

A convenient device for dealing with expressions of this kind makes use of the principle that, if both numerator and denominator of a fraction are multiplied by the same quantity, the value of the fraction is unchanged. In this example, we multiply both numerator and denominator by 42, obtaining

$$\frac{6 + 7}{42 - 1} = \frac{13}{41}$$

Example 2. Simplify the expression

$$\frac{\frac{9s}{1 - 27s^3} + \frac{3}{3s - 1}}{1 + \frac{1 - 18s^3}{27s^3 - 1}}$$

The quantity by which the numerator and denominator are multiplied is chosen to be a multiple of the denominators of each part. Since

$$27s^3 - 1 = (3s - 1)(9s^2 + 3s + 1)$$

we see that the least common multiple of the denominators of the parts is $27s^3 - 1$. Observe that

$$\frac{27s^3 - 1}{1 - 27s^3} = -1$$

Hence the given expression reduces to

$$\begin{aligned} & \frac{(9s)(-1) + (3)(9s^2 + 3s + 1)}{(27s^3 - 1) + (1 - 18s^3)} \\ &= \frac{27s^2 + 3}{9s^3} \\ &= \frac{9s^2 + 1}{3s^3} \end{aligned}$$

Exercises

1. Find the factors of:

- (a) $p^2 - p^2q^2$
- (b) $s^2 - 5s + 6$
- (c) $2y^2 + y - 6$
- (d) $8R^3 - 27$
- (e) $\sin^2 A - \cos^2 A$
- (f) $1 + \tan^2 A$
- (g) $6rt - 8pt - 4p + 3r$
- (h) $x^2 - (a - k)x - ak$

2. Simplify the expressions:

$$(a) \frac{\frac{P+Q}{Q} - \frac{P+Q}{P}}{\frac{1}{Q} - \frac{1}{P}}$$

$$(b) \frac{1}{1 + \frac{1}{1 + \frac{1}{n}}}$$

$$(c) \frac{R - T + \frac{T^2}{R}}{\frac{T}{R^2} + \frac{R}{T^2}}$$

$$Ans. \quad \frac{RT^2}{R+T}$$

3. If $f(x) = \frac{1}{1-x}$, what is the value of $f(0)$? Simplify the complex fraction $f\left(\frac{2}{n+1}\right)$.

4. If $f(x) = \frac{1}{x - \tan x}$, what is the value of $f\left(\frac{\pi}{3}\right)$? Ans. -1.46

5. Find by division the simplest form for the fraction

$$\frac{x^3 - x^2 - 7x + 15}{x + 3}$$

6. Find by division the simplest form for the fraction

$$\frac{2y^4 - 3y^3 - 9y^2 + 15y - 5}{2y^2 - 3y + 1}$$

7. If $f(x) = \frac{x}{1-x}$, show that

$$f(x) - f(y) = \frac{x-y}{xy} f(x)f(y)$$

8. Express in your own words the meaning of equations 3-2, 3-3, and 3-4.

9. If $f(m) = \frac{1+2m}{m(1+m)}$, find $f\left(\frac{1}{a-1}\right)$ and simplify.

$$\text{Ans. } \frac{a^2-1}{a}$$

10. If $F(A) = \cos^2 A$, find $F(x) + F(90^\circ - x)$ and simplify.

11. Simplify the expression

$$\frac{\frac{2}{3m-1} - \frac{6m}{9m^2-1}}{\frac{3m}{1-9m^2} + \frac{3+m}{m+3m^2}}$$

$$\text{Ans. } \frac{2m}{8m-3}$$

12. If $F(z) = \frac{z+1}{z-1}$, find $F\left(\frac{1-z}{1+z}\right)$ and simplify.

13. Factor $10 - 16R - 8R^2$.

14. Factor $8py - 4p + 3q - 6qy$.

15. Simplify the expression

$$\frac{\sin A + \cos A}{\csc A + \sec A}$$

$$\text{Ans. } \sin A \cos A$$

16. Simplify the expression

$$\sin B + \frac{\sin^2 B \sec B}{1 + \sec B}$$

17. Divide $\frac{1-ar-(a+r)(-a)}{(1-ar)^2}$ by $1 + \left(\frac{a+r}{1-ar}\right)^2$, and reduce to simplest form.

$$\text{Ans. } \frac{1}{1+r^2}$$

35. The Sigma Notation. In the development of the language of algebra, two objects have always been sought: generality and conciseness. Many formulas can be concisely expressed by means of the sigma notation. The symbol \sum (Greek letter capital sigma) may be translated *sum of terms of the form*—. For example,

$$\sum_{i=1}^n a_i x^i = a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

The letter i is called the index of summation. The initial and final values of the index are shown below and above the operator \sum . The first term in the sum is that for which i takes on the value shown below the sign of summation. In the second term of the sum, the value of i is increased

by 1; in the third term, i increases again by 1, and so on. In the last term of the sum, i takes on the value shown above the sign of summation. In the above example, the expression ax^i is called the general term of the sum, because it represents any term.

The reference to \sum as an *operator* suggests a point of view towards algebra which, although not new to mathematicians, has only in recent years gained widespread acceptance and understanding among the users of mathematics. Algebraic expressions are *directions to perform certain operations*. Thus the expression $1/N$ can be regarded as directing us to perform the operation of taking the reciprocal of a number N . Again, we may regard the expression $\sin A$ as directing us to perform the operation of finding the sine of an angle A . The operator here is the symbol *sin*. (In passing, it might be mentioned that it is bad form to write formulas such as

$$\tan = \frac{\sin}{\cos}$$

unless, of course, one wishes to emphasize the operational significance of the tangent as the quotient of two operators!) It is sometimes helpful, when one is trying to grasp the meaning of a complex algebraic expression, to ask, *What is the first, or principal, operation that I am directed to perform?*

Example. Evaluate the expression $\sum_{i=1}^4 \frac{(r-b)^i}{2i}$ for $r = 5$ and $b = 1$.

The value of the expression will be the same, whether the numerical values of r and b are substituted before summing, or after. The calculation is easier if $\frac{(r-b)^i}{2i}$ is simplified before summing.

$$\begin{aligned} \sum_{i=1}^4 \frac{(5-1)^i}{2i} &= \sum_{i=1}^4 \frac{4^i}{2i} \\ &= \frac{4^1}{2(1)} + \frac{4^2}{2(2)} + \frac{4^3}{2(3)} + \frac{4^4}{2(4)} \\ &= \frac{146}{3} \end{aligned}$$

Exercises

1. Write in expanded form $\sum_{i=1}^4 2x^i$.
2. Write in expanded form $\sum_{i=1}^4 (2x)^i$.

3. Write in expanded form $\sum_{i=1}^3 ix^{i+1}$.
4. Write in expanded form $\sum_{i=1}^n ar^i$.
5. Write in summation notation the expression

$$4A^4 + 8A^5 + 16A^6 + 32A^7$$
6. Write in summation notation the expression

$$2x^2 + 4x^3 + 6x^4 + 8x^5$$
7. Write in summation notation the expression

$$(x+3)^2 + (x+6)^3 + (x+9)^4$$
8. Write in expanded form $\sum_{n=1}^4 \frac{(n-1)^2}{n(n+1)}$ and evaluate.
9. If $F(n) = \sum_{i=1}^3 (2n)^{i-1}$, find $F(3)$ and $F(a-1)$, and simplify.
- Ans. $43, 4a^2 - 6a + 3$
10. When an object is weighed several times on a sensitive laboratory balance, the values obtained are usually found to differ slightly from one another. The formula $\bar{w} = \frac{1}{n} \sum_{i=1}^n w_i$ is usually employed in calculating the "best value" for the weight. Express in words the meaning of this formula.
11. Write in expanded form the formula for calculating the center of gravity of an object whose parts have the weights w_1, w_2, \dots, w_n :

$$\bar{x} = \frac{\sum x_i w_i}{\sum w_i} \quad (i = 1, 2, \dots, n)$$

Is it legitimate to "cancel" the two sigma signs? The two w 's?

CHAPTER 6

THE LINEAR FUNCTION

36. Equations. A fundamental problem of algebra is solving equations. It is a problem of the utmost importance, since success in finding the answer to technical problems often hinges on the solution of one or more equations.

Some preliminary remarks on the use of the equality sign are in order. The symbol $=$ is a conspicuous exception to the rule that algebraic symbols have an unambiguous and precisely defined meaning. The two principal meanings are illustrated in the following examples:

$$\begin{aligned}kx - 3 &= 4a^2 \\ a^2 - x^2 &= (a + x)(a - x)\end{aligned}$$

The first equation expresses a restriction upon possible values of x (assuming that values of a and k are assigned in advance). It implies that x must have the value $\frac{4a^2 + 3}{k}$. It may be called a *condition upon* x , or a conditional equation.

The second equation implies no restriction upon either x or a . The left-hand member is *identically equal* to the right-hand member, which means that the equality holds for all possible values of x and a . The two members differ only in form, never in value. Either expression may replace the other whenever convenient; and this statement is true, not only when a and x represent numbers, but also when they represent complicated mathematical expressions, so long as the latter obey the laws of algebra. In order to emphasize the stronger meaning, it is permissible to use the symbol \equiv ; but experience has shown that the ambiguity of meaning of the ordinary sign is convenient in practice, and that confusion rarely arises, even on the part of apprentices in the study of mathematics, after the distinction in meaning has been grasped.

By *solving an equation* is meant the process of finding those values of the unknown quantity for which the equation is true. These values are called the roots of the equation. In this course, we shall study the simpler kinds of equations, which fortunately are also the commonest in technical work, in order to discover appropriate methods for their solution.

Exercises

1. The law of Gay-Lussac expresses the volume v of an ideal gas at a temperature t on the Centigrade scale by the formula $v = v_0 \left(1 + \frac{t}{273} \right)$. When $t = 0$, $v = v_0$. Hence v_0 represents the volume at 0°C . Is this a conditional equation, or an identity?
2. Give three examples of identities, and three conditional equations, which you have encountered. Find an additional example of each kind of equation in Chapter 1 of this book.

37. Equations of the First Degree. Equations of this kind are briefly described by saying that they are of the form

$$ax + b = 0$$

This implies that there may be just two kinds of terms in an equation of the first degree; (a) terms in which the unknown does not occur at all; and (b) terms in which the unknown occurs, but only as a factor raised to the first power.

In the language of algebra, the solution is expressed as follows:

$$\begin{aligned} ax &= -b \\ x &= -\frac{b}{a} \end{aligned}$$

Translating into English, we see that the solution is obtained by rearranging and transposing terms until those containing the unknown are in the left-hand member, and all other terms are in the right-hand member. The final step is to divide both members by the coefficient of the unknown.

Equations with numerical coefficients are checked by direct substitution in the original equation. Equations with literal coefficients are often difficult to check by direct substitution. It may be wiser to select suitable numerical values for the literal coefficients, calculate the corresponding value of the unknown from the literal solution, and then test these values in the original equation.

Example. It is required to solve the following equation for y' in terms of the other letters:

$$2ax + bxy' + by + 2cy' = 0$$

Subtracting the expression $2ax + by$ from both members, we have

$$(bx + 2cy)y' = -2ax - by$$

Hence

$$y' = -\frac{2ax + by}{2cy + bx}$$

Taking $a = +2$, $b = -5$, $c = -2$, $x = -3$, and $y = +3$, the solution yields $+9$ for the value of y' . Checking these numbers in the original equation:

$$\begin{aligned}-12 + 135 - 15 - 108 &= 0 \\ 135 - 135 &= 0\end{aligned}$$

Exercises

Solve and check the following equations.

1. $0.05(0.81R + 3.12) = 0.75(2.26R - 1.77)$

2. $0.46x - 1.52(x - 0.24) = 5.64 + 0.83x$

3. $\frac{2}{m-3} + 4 = \frac{5}{3-m}$

4. $\frac{4}{x-2} + \frac{3x}{x^2-3x+2} = \frac{2}{1-x}$

5. Solve for y_2 in terms of the other letters:

$$\bar{y} \sum_{i=1}^3 P_i = \sum_{i=1}^3 y_i P_i$$

6. Solve for R_2 in terms of the other letters:

$$\frac{1}{R} = \sum_{i=1}^3 \frac{1}{R_i}$$

7. Solve for x in terms of the other letters:

$$\frac{x-b}{h-y} = \frac{a-b}{h}$$

8. Solve the equation

$$S = \frac{rL - a}{r - 1}$$

for r , in terms of the other letters. Check by substituting your answer in the original equation and reducing both members to an identical form.

9. Solve the equation

$$S = \frac{Ht_2}{m(t_2 - t_1)}$$

for t_2 in terms of the other letters. Check by substituting your answer in the original equation.

10. Solve the equation

$$g = g_0 \left(1 - \frac{2h}{R} \right)$$

for R in terms of the other letters. Check by substituting your answer in the original equation.

11. Solve the equation

$$C = K \left(\frac{rr'}{r - r'} \right)$$

for r' in terms of the other letters. Check by substituting your answer in the original equation.

38. Problem Solving. Skill in the formal operations of algebra is of small value to an engineer, unless he is able to translate his professional problems into the language of algebra when necessary. The prospective engineer must therefore take care that a bridge be built between the study of algebra and the study of professional engineering subjects.

When difficulty is experienced in setting up a problem in algebraic form, the following suggestions may help the student in getting started.

(a) Pick out the thing or things which are to be found. Assign letters to represent the unknown quantities. State clearly just what each letter stands for, giving units.

(b) Look for a physical or geometrical law or principle relevant to the problem, such as the principle of conservation of mass, or the geometrical axiom that the whole equals the sum of its parts.

(c) Make a statement in words, showing how the law or principle applies to this particular problem.

(d) Translate this statement into the language of algebra. This step typically leads to an equation. If it contains more than one unknown, additional equations will usually be needed. They are obtained in a similar way.

The solution to most problems may be analyzed into these four elements, as is shown in the following examples; but in the operation of a trained mind, no such rigid pattern as this is likely to be followed. The analysis does not represent the end-product of training; it is intended only to aid the young engineer in understanding the manner in which algebra is applied to technical problems.

Example 1. How many milliliters of 85% hydrochloric acid must be added to M milliliters of a solution containing 1.2% of the acid, in order to obtain a 3.6% (normal) solution?

Let x ml. be the amount of 85% acid required. The principle of conservation of mass can be applied: The amount of acid in the original solution plus the amount added gives the total amount in the new solution. But

$$0.012M = \text{ml. of acid in original solution}$$

$$0.85x = \text{ml. of acid added}$$

$$0.036(M + x) = \text{ml. of acid in new solution}$$

The equation is therefore

$$0.012M + 0.85x = 0.036(M + x)$$

$$x = 0.029M \text{ milliliters.}$$

Example 2. In order to determine the specific gravity of a new plastic material, a piece of the plastic is weighed while immersed in water. Its weight in water is W_w ; its weight in air is W_a . Find the specific gravity.

Let x be the specific gravity of the unknown. By definition, the specific gravity is the weight of a certain volume of the unknown, divided by the weight of an equal volume of water.

The physical principle which is applicable here is Archimedes' principle: a body immersed in a fluid is buoyed up by a force equal to the weight of displaced water. This buoyant force causes an apparent loss of weight when the object is weighed while immersed in water. That is,

$$\begin{aligned}\text{Weight of displaced water} &= \text{Apparent loss of weight} \\ &= W_a - W_w\end{aligned}$$

Hence

$$\begin{aligned}x &= \frac{\text{weight of body}}{\text{wt. of water displaced by body}} \\ &= \frac{W_a}{W_a - W_w}\end{aligned}$$

Exercises

1. At what time between 3 and 4 o'clock will the hands of a clock be together?
2. At what time between 7 and 8 o'clock will the hands of a clock be together?
3. If the radiator of an automobile holds 10 quarts of a water solution containing 15% antifreeze, how much of the solution must be drawn off and replaced with pure antifreeze, in order to bring the percentage of antifreeze up to 35%?
4. If the radiator of an automobile holds Q quarts of a water solution containing $p\%$ antifreeze, how much of the solution must be drawn off and replaced with pure antifreeze, in order to bring the percentage of antifreeze up to $s\%$?

$$\text{Ans. } \left(\frac{s - p}{100 - p} \right) Q$$

5. How many pounds of water must be evaporated from 95 lb. of a solution that is 3.0% salt, in order to obtain a solution that is 7.0% salt? Ans. 54 lb.
6. It is claimed that a new set of spark plugs, costing \$6.50, will increase the mileage of a certain car from 14 to 17 miles per gallon. With gasoline at 21 cents per gallon, how many miles must be driven before the saving will pay for the new plugs?
7. How many pounds of pure copper must be added to 165 pounds of an alloy containing 8.0% copper, in order to obtain an alloy that is 12.0% copper?

$$\text{Ans. } 7.5 \text{ lb.}$$

8. The Boston branch of a certain concern shows gross profits of \$99,000 for the year. The manager gets a bonus of 5% of the net profit, computed after the income tax is paid. The income tax is 20% of the profits, after the bonus has been deducted. What bonus does the manager get?
9. Find a formula for the bonus in Exercise 8, if gross profits are P , the income tax rate is i , and the bonus percentage is b .

10. According to tradition, Archimedes discovered the principle named for him while considering the following problem: A crown fashioned of gold contained an unknown percentage of silver alloyed with the gold. Take the weight of the crown as 642 grams in air, and 603 grams in water. Find the percentage of silver. It may be assumed that the volume of the alloy is equal to the sum of the volumes of gold and silver of which it is composed. (The specific gravity of gold is 19.3, and of silver, 10.6.)

39. Graph of the Linear Function. The theory of equations of the first degree is a part of the theory of the first degree function. All functions of the first degree are represented by the expression

$$f(x) = mx + b \quad 6-1$$

It will be shown in Chapter 16 that the graph of equation 6-1 is a straight line; hence in plotting it, only two points need be computed. It is well, however, to plot a third point, as a check.

The graphical solution of equations of the first degree consists in finding the value of x for which $f(x) = 0$; which is equivalent to finding the point where the line crosses the x -axis.

The linear function is often used in constructing nomographic charts, which are used by engineers for solving certain kinds of equations graphically. For example, suppose we wish to express 27° Fahrenheit on the Centigrade scale.

It is easy to solve this *one* problem by substituting the given data in the equation

$$F = \frac{9}{5}C + 32 \quad 6-2$$

But if many such conversions are necessary, it saves time to graph the linear function 6-2, as Figure 31.

Table

C	F
0	32
10	50
20	68

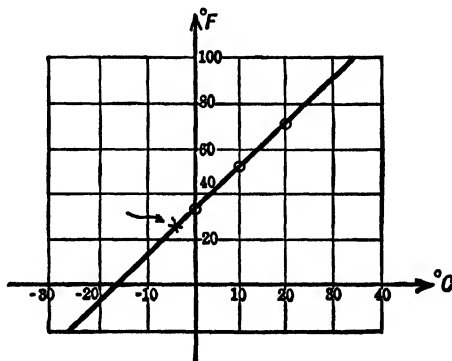


FIG. 31

The Fahrenheit temperatures corresponding to 0° , 10° , and 20° Centigrade are tabulated, and three points on the graph of the linear function are plotted. From the graph, it is evident that 27° F. corresponds to -3° C., approximately.

40. Direct Variation. A special case of the linear function is represented by the expression

$$y = kx \quad 6-3$$

which may be translated, *y is directly proportional to x; or y varies directly as x.* The coefficient k is called the constant of proportionality.

Although equation 6-3 has been called a special case of the linear function, it is always possible to express a relation of the form 6-1 in the form 6-3 by changing the zero point of the scale on which x is measured. For example, the law of Gay-Lussac is usually written

$$v_t = v_0 \left(1 + \frac{t}{273} \right) \quad 6-4$$

Here v_t represents the volume of an ideal gas at a temperature t on the Centigrade scale; and v_0 represents the volume at zero on the Centigrade scale. If we set

$$t = T - 273$$

the law of Gay-Lussac takes the form

$$v_t = \frac{v_0 T}{273} \quad 6-5$$

That is, the volume is directly proportional to the absolute temperature.

Again, it is always possible to express a relation like 6-3 in the form of a proportion. Let V_1 represent the volume of equation 6-5 at temperature T_1 , and, similarly, let V_2 represent the volume at T_2 . We have then

$$V_1 = \frac{v_0}{273} T_1$$

$$V_2 = \frac{v_0}{273} T_2$$

Dividing each member of the first equation by the corresponding member of the second, we obtain the proportion

$$\frac{V_1}{V_2} = \frac{T_1}{T_2} \quad 6-6$$

This last form is convenient for slide rule calculations.

Exercises

- Express equation 2-4 in proportion form, taking r as the constant of proportionality, and translate into English.
- Express equation 2-4 in proportion form, taking θ as the constant of proportionality, and translate into English.
- Assuming a convenient value for v_0 , represent Gay-Lussac's law (equation 6-4) by means of a graph. Does the graph extend below the t -axis?
- Discuss the meaning of the equality sign in equation 6-1.
- If a steel wire is stretched by the application of a force P , the length is given by Hooke's law:

$$L = L_0(1 + \alpha P)$$

Represent Hooke's law by means of a graph. Does the graph extend to the left of the L -axis?

- Express Hooke's law in the form of a proportion.
- Express Hooke's law in the form of equation 6-3. What physical quantity is represented by y in this form?

Miscellaneous Problems

- If $F(x) = \frac{2-x^3}{4}$, find

$$F(2)$$

$$F(-1)$$

$$F(0)$$

$$F\left(\frac{1}{x}\right)$$

$$F(a^2)$$

$$F(2-x)$$

- Factor the following expressions:

$$\sin^2 A - 7 \sin A - 44$$

$$6R^2 + RT - 12T^2$$

$$64 + \tan^3 \theta$$

$$x^2 + px - 2nx - 2np$$

- Express the volume of a hollow cylinder of height h and radii of base R and r as a function of h , R , and r in factored form.
- One leg of a right triangle is s and the other is 7. Express the length of the hypotenuse as a function of s .
- An automobile is traveling at the constant speed of r miles per hour. How many minutes will be required to cover d miles?
- If $\cos A = \frac{4}{5}$, and A is an acute angle, find the other five functions of A in radical form.
- Reduce the expression

$$\frac{2 \cos A + \cot A}{\csc A - \sin A}$$

to the form $\sec A + 2 \tan A$.

8. What is the area of a sector of a circle of radius 18 inches, if the angle of the sector is 1.75 radians?
9. Express the circumference of a circle as a function of its area.
10. Simplify $\frac{5}{x^3 + 1} + \frac{2}{x - 1 - x^2}$
11. State whether each of the following statements is true or false. If false, point out the error.

$$(x + 2y)^2 = x^2 + 4y^2$$

$$\frac{6x - 8}{2x - 1} = \frac{3x - 4}{x - 1}$$

$$\frac{ax + az}{x(x + z)} = \frac{a}{x}$$

$$\frac{R + T}{3 - x} = \frac{R + T}{x - 3}$$

$$(s - b)^2 = (b - s)^2$$

$$\frac{1}{a + b} + \frac{1}{c} = \frac{1}{a + b + c}$$

$$\text{If } F(x) = \frac{a + x}{a - x}, \quad F(a) = 0$$

12. Simplify the expression

$$\frac{1 - r_2^2}{r_1^2 + r_1} \left(-\frac{1 - r_1^2}{1 + r_2} \right) \left(1 + \frac{r_1}{1 - r_1} \right)$$

Explain why it is not a satisfactory check on the work to take $r_1 = 1$, or $r_1 = -1$.

13. Express in the summation notation the sum of the first one hundred of all the numbers which are divisible by 7.
14. Derive a formula for finding the area of the cross-section of the channel beam shown in Figure 32.
15. Simplify the expression

$$\frac{\frac{a - 4}{a^3 - 1} - \frac{1}{1 - a}}{1 - \frac{a^2}{a^2 + a + 1}}$$

16. How many pounds of nickel must be added to b pounds of an alloy that is 14% nickel, in order to obtain an alloy that is 23% nickel?

17. Solve the equation

$$F = \frac{f_1 f_2}{f_1 + f_2 - d}$$

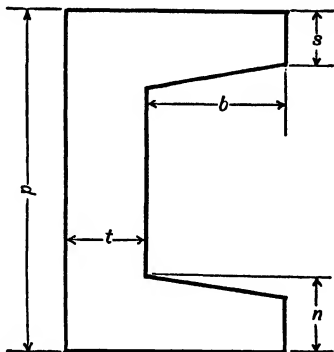


FIG. 32

for f_2 in terms of the other letters. Check by substituting your answer in the original equation.

18. If $C = \frac{N}{R + \frac{R}{K}}$, does C increase or decrease when K increases? (N , R , and K are all

positive.)

19. Express in simplest form

$$\frac{\frac{a}{b}}{c} \quad \text{and} \quad \frac{a}{\frac{b}{c}}$$

20. In manufacturing a certain type of lock there is an overhead cost of \$10,000 which must be paid whatever the number of locks made. In addition, it costs 11 cents to make each lock plus 18 cents per pound for the material used. Denote by n the number of locks made, and by w the weight, in pounds, of each lock. Express the cost C of one lock as a function of n and w .
21. If $\tan A - \cot B = 0$, show that $\sec A = \csc B$, if A and B are acute angles.
22. Factor $x^2 - 4xy - y^2 - 4a^2$. Also factor $a^4 + 4x^4$ by adding and subtracting a term that will transform the given expression into the difference of two squares.
23. A man starts from a point 5 miles directly east of a certain place and travels north at the rate of 4 miles per hour. What function expresses his distance from the place at the end of t hours?
24. The deflection at the end of a cantilever beam of length L , with a concentrated load at any point, is

$$y = -\left(\frac{P}{EI}\right)\left(\frac{b^3}{3} + \frac{ab^2}{2}\right)$$

Set $a = b = \frac{L}{2}$, and simplify the resulting expression. (This gives the deflection for a load at the mid-point of the beam.)

25. (a) The volume of a spherical shell, whose outer radius is R , and whose inner radius is r , is

$$\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$$

The surface area of a sphere, whose radius lies halfway between R and r , is

$$4\pi\left(\frac{R+r}{2}\right)^2$$

An approximate formula for the volume of a spherical shell is obtained by multiplying the surface area of a sphere, whose radius lies halfway between the inner and outer radii of the shell, by the thickness of the shell. The error committed by using the approximate formula is

$$\left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3\right) - 4\pi\left(\frac{R+r}{2}\right)^2(R-r)$$

Simplify this formula for the error, by factoring.

(b) The radius of the earth is about 4000 miles. Its outside layer of silica rocks is about 500 miles thick. What will be the error in a calculation of the volume (in cubic miles) of this outer shell, if the thickness of the shell is multiplied by the surface area of a sphere of radius 3750 miles?

26. Factor $a^{12} + 64$.

27. At a concert the price of tickets was 85 cents for adults, and 55 cents for students under eighteen. The ticket seller found that he had sold 637 tickets for \$415.15. How many student admissions were there?

28. The tensile unit-stress of a riveted plate is given by

$$s = \frac{P}{(p - d)t}$$

Solve for p in terms of the other letters.

29. Solve the formula

$$\frac{1}{D_i} + \frac{1}{D_o} = \frac{1}{f}$$

for D_i in terms of the other letters.

30. Simplify

$$\frac{k^2 - h^2 + \left(\frac{k^2}{h} - h\right)^2}{\frac{k^2}{h} - h}$$

(This expression occurs in discussing the center of oscillation of a compound pendulum.)

31. (a) If $\tan \theta = \frac{L\omega}{R}$, find a formula for $\cos \theta$, in terms of R , L , and ω .

(b) Given that $I = \frac{E}{\sqrt{R^2 + L^2\omega^2}}$, express $\cos \theta$ in terms of I , R , and E .

32. Express the altitude of a trapezoid as a function of its bases B and b , and its area A .

33. The net stress in plates for self-sustaining steel stacks is

$$S = \frac{W}{\frac{\pi}{4}(d_1^2 - d_2^2)} + \frac{Ph}{\frac{\pi}{32}\left(\frac{d_1^4 - d_2^4}{d_1}\right)}$$

By substituting the expressions

$$q = \frac{d_1^3 + d_2^3}{8d_1}$$

$$\frac{I}{e} = \frac{\pi(d_1^4 - d_2^4)}{32d_1}$$

reduce the equation for the stress to the form

$$S = \frac{Wq + Ph}{\frac{I}{e}}$$

Suggestion: Multiply the first term in the right-hand member of the original equation for S by the quantity

$$\frac{q}{d_1^2 + d_2^2} \frac{1}{8d_1}$$

34. The deflection in reinforced concrete beams is given by

$$f = K \frac{L^3}{d} (C_c - C_s)$$

Solve for C_c in terms of the other letters. (C_c and C_s are the unit deformations in extreme fibers of concrete and steel, respectively. L is the length and d the depth of the beam.)

35. Solve the equation

$$S = \frac{P}{a} \left(1 + \frac{4e}{d} \right)$$

for d in terms of the other letters. (This formula gives the unit stress in a helical spring, where d is the diameter of the spring rod.)

36. In the design of a triple-riveted, double strap butt joint for a boiler, the thickness of the strap plate is found from the equation

$$(p - 2d)tS_t + d'S_b = (p - d)tS_t$$

Solve for t' in terms of the other letters. (The result shows the thickness to be independent of the pitch p and the rivet hole diameter d . It is directly proportional to the thickness of the shell and to the unit tensile stress.)

37. Solve the equation

$$R = \frac{Wh}{1 + \frac{A\sqrt{W}}{27}}$$

for h in terms of the other letters. (This formula gives the amount of radiant heat absorbed in coal-fired, water-tube boilers.)

38. Solve the equation

$$e = 1 - K \left(\frac{T_d - T_b}{T_c - T_b} \right)$$

for T_c in terms of the other letters. (This is the formula for the efficiency of a Le Noir gas engine. T_c is the ignition temperature.)

39. From the equation

$$c_p(T_2 - T_1) = c_v(T_2 - T_1) + \frac{R(T_2 - T_1)}{778}$$

show that the ratio $\frac{c_p}{c_v}$ is equal to

$$\frac{1}{1 - \frac{R}{778c_p}}$$

(The ratio of the specific heat at constant pressure, c_p , to the specific heat at constant volume, c_v , is important in the thermodynamics of a gas. The result indicates that the ratio is independent of the change in temperature, and that it may be calculated from the easily measured specific heat at constant pressure, since the gas constant, R , is known.)

CHAPTER 7

SIMULTANEOUS LINEAR EQUATIONS

41. Two Unknowns. When several quantities in a problem are unknown, we are led to consider a set of equations which must be satisfied simultaneously. For two equations, of the first degree in each unknown, the general case is

$$\begin{cases} a_1x + b_1y = c_1 & 7-1 \\ a_2x + b_2y = c_2 & 7-2 \end{cases}$$

These equations may be solved by various devices. One of these is called the method of *elimination by substitution*. Though unpretentious, it is straight-forward and well adapted to engineering problems, whose coefficients are often awkward decimal quantities. One variable, say y , is selected for elimination. From equation 7-1, we find

$$y = \frac{1}{b_1} (c_1 - a_1x) \quad 7-3$$

Substituting this expression in equation 7-2, we have

$$a_2x + \frac{b_2}{b_1} (c_1 - a_1x) = c_2$$

From this equation we easily obtain

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad 7-4$$

Equation 7-3 now yields

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \quad 7-5$$

By the use of determinants, presently to be explained, these results may be written in a more easily remembered form. In solving technical problems, it is usually simpler to use the method than the formulas.

Another method for solving simultaneous equations is called *elimination by addition or subtraction*. If both members of equation 7-1 be

multiplied by b_2 , and both members of equation 7-2 by b_1 , the result is

$$a_1b_2x + b_1b_2y = c_1b_2$$

$$a_2b_1x + b_1b_2y = c_2b_1$$

If now the second equation be subtracted from the first, we have

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1$$

which leads at once to equation 7-4.

A third method of solution is to graph the linear functions 7-1 and 7-2. The coordinates of the point of intersection of the two straight lines satisfy both equations simultaneously, and hence constitute the solution.

Example. A problem in mechanics leads to the simultaneous equations

$$\begin{cases} P \cos 30^\circ - 0.40N \cos 30^\circ - N \sin 30^\circ = 0 \\ P \sin 30^\circ - 0.40N \sin 30^\circ + N \cos 30^\circ = 32 \end{cases}$$

It is required to find the value of P satisfying these equations.

They are first written in standard form:

$$\begin{cases} 0.866P - 0.846N = 0 \\ 0.500P + 0.666N = 32 \end{cases}$$

In order to eliminate N , let the first equation be solved for N in terms of P :

$$N = \frac{0.866}{0.846}P = 1.024P$$

Substituting in the remaining equation, we have

$$0.500P + 0.666(1.024P) = 32$$

$$P = 27$$

(The answer has been rounded off.)

Figure 33 shows the same problem solved graphically.

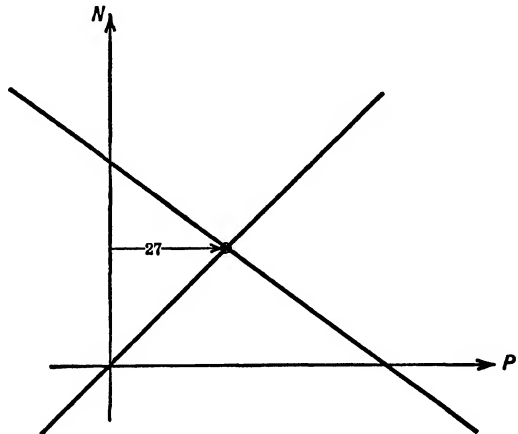


FIG. 33

Exercises

Solve and check the following pairs of simultaneous equations.

$$1. \begin{cases} 1.2x - 2.3y = 5.7 \\ 0.71x + 1.15y = 0.41 \end{cases}$$

$$2. \begin{cases} 0.14x + 2.7y = 0.756 \\ 2.7x - 0.14y = 7.27 \end{cases}$$

$$3. \begin{cases} (a-b)x + by = a^2 + b^2 \\ 2bx = (a+b)y \end{cases}$$

$$4. \begin{cases} 2x + (a-b)y = (a+b)^2 \\ ax - 2b^2y = 2b^3 \end{cases}$$

$$5. \begin{cases} \frac{5}{R_1} - \frac{2}{R_2} = 7 \\ \frac{2}{R_1} - \frac{3}{R_2} = 5 \end{cases}$$

$$6. \begin{cases} \frac{1}{L} + \frac{1}{M} = \frac{1}{a} \\ \frac{1}{L} - \frac{1}{M} = \frac{1}{b} \end{cases}$$

42. Three or More Unknowns. When three unknown quantities occur in a problem, a set of three independent equations is usually employed in order to obtain them. The significance of the word *independent* will be discussed presently.

Graphical solutions are not practicable when there are three or more unknowns. The method of substitution is recommended for numerical problems, especially those containing non-integral coefficients.

Example 1. A problem in mechanics leads to the following equations:

$$\begin{cases} \frac{4}{3}D + \frac{3}{5}E = 2340 \\ A_s + \frac{3}{5}D - \frac{4}{5}E = 500 \\ 14.4A_s + 6.4(\frac{3}{5}D) - 6.4(\frac{4}{5}E) = 0 \end{cases}$$

From the first of these equations, we find

$$D = 2920 - 0.75E \quad 7-6$$

Substituting this expression in the second and third equations, we have

$$A_s + \frac{3}{5}(2920 - 0.75E) - \frac{4}{5}E = 500$$

$$14.4A_s + \frac{19.2}{5}(2920 - 0.75E) - \frac{25.6}{5}E = 0$$

Simplifying these equations, the result is

$$\begin{cases} A_s - 1.25E = -1250 \\ 14.4A_s - 8E = -11,200 \end{cases}$$

The first of these two equations gives

$$A_s = 1.25E - 1250 \quad 7-7$$

Substituting this expression in the second equation, and solving, we have

$$E = 680$$

The equation of substitution 7-7 now yields

$$A_s = -400$$

The equation of substitution 7-6 leads to

$$D = 2410$$

Example 2. For many substances, the specific heat is given by a formula of the form

$$c_p = a + bT + cT^2 \quad 7-8$$

Here T represents the absolute temperature, and a , b , and c are constants which must be determined for each substance. The following table shows the specific heat of graphite at three different temperatures:

T	c_p
278	2.49
517	3.45
704	4.11

Since each pair of values may be assumed to satisfy equation 7-8, we find that a , b , and c must satisfy the following equations:

$$\begin{cases} a + 278b + 77,300c = 2.49 \\ a + 517b + 267,000c = 3.45 \\ a + 704b + 495,000c = 4.11 \end{cases}$$

Let us solve the first equation for a :

$$a = 2.49 - 278b - 77,300c \quad 7-9$$

Substituting this expression in the second and third equations, and simplifying, we have

$$\begin{cases} 239b + 190,000c = 0.96 \\ 426b + 418,000c = 1.62 \end{cases}$$

The first of these two equations gives

$$b = 0.004\ 02 - 795c \quad 7-10$$

Substituting this expression in the second equation, and solving for c , we have

$$c = -0.000\ 001\ 1$$

The equation of substitution 7-10 now yields

$$b = 0.004\ 89$$

The equation of substitution 7-9 leads to

$$a = 1.22$$

Hence the specific heat formula for graphite is

$$c_p = 1.22 + 0.004\ 89T - 0.000\ 001\ 1T^2$$

Exercises

$$1. \begin{cases} 2x_1 - x_2 - 3x_3 = 1 \\ 4x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 5x_2 + 7x_3 = -3 \end{cases} \quad 2. \begin{cases} 5L + 2M + 2N = 1 \\ 2L - 4M + 5N = 1 \\ L + 8M - 3N = 9 \end{cases}$$

3. Determine the constants a , b , c in the equation $y = ax^2 + bx + c$, given the table

x	y
2	17
3	-33
5	-229

4. Determine the constants D , E , F in the equation $x^2 + y^2 + Dx + Ey + F = 0$, given the table

x	y
-2	+6
+1	-3
+5	+5

$$5. \begin{cases} 4i_1 + 3i_2 = 1 \\ 5i_2 - 4i_3 = -5 \\ i_3 + 3i_4 = 2 \\ 7i_4 + 2i_1 = -11 \end{cases} \quad 6. \begin{cases} i_1 + i_2 + i_3 = -5 \\ i_2 - i_3 + i_4 = 0 \\ i_3 - i_4 + i_1 = -7 \\ i_4 - i_1 - i_2 = +12 \end{cases}$$

43. Determinants. For theoretical investigations of problems involving simultaneous equations, a powerful tool exists in the theory of determinants. The determinant of the second order is defined by the equation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \quad 7-11$$

The quantities a_1 , a_2 , b_1 , b_2 are called *elements* of the determinant. Thus, a determinant with two rows and two columns is evaluated by multiplying together the element in the upper left-hand corner and the element diagonally opposite it, and subtracting the product of the element in the upper right-hand corner and the element diagonally opposite it.

A solution for the simultaneous equations

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has been obtained (equations 7-4 and 7-5) This solution may now be written in the form

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{D} \quad 7-12$$

where D stands for the determinant 7-11. The determinant in the numerator of the expression for x is obtained from D by replacing the coefficients of x by the constant terms. The determinant in the numerator of the expression for y is obtained from D by replacing the coefficients of y by the constant terms. D itself is formed from the coefficients of the unknowns.

If it should happen that the value of D is zero, we see from equation 7-11 that

$$a_2 b_1 = a_1 b_2$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

That is, the coefficients of x and y in equation 7-2 are directly proportional to those in equation 7-1. If D vanishes, the solution 7-12 fails; the equations are said to be **inconsistent**. If all three of the determinants in the solution 7-12 vanish, it is easily seen that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

The equations are **not independent**.

Turning to linear equations in three unknowns, we may study the forms

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases} \quad 7-13$$

The coefficients of the unknowns form a determinant of the third order, whose value is given by the equation

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{aligned} &a_1 b_2 c_3 - a_1 b_3 c_2 \\ &+ a_2 b_3 c_1 - a_2 b_1 c_3 \\ &+ a_3 b_1 c_2 - a_3 b_2 c_1 \end{aligned} \quad 7-14$$

This formula is easily remembered with the aid of the cyclic scheme



When the subscripts occur in cyclic order, the sign of the term is +; when the cyclic order is reversed, the sign is -. Each of the six products in equation 7-14 contains just one element from each row and from each column of the determinant; and just six distinct products can be obtained by selecting one element from each row and column.

The solutions to equations 7-13 may now be written down in the form

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D} \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D} \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D} \quad 7-15$$

The verification of this statement by direct substitution in equations 7-13 is not difficult, though tedious. Observe that the determinant in the numerator of the expression for any unknown is obtained from D by replacing the coefficients of that unknown by the constant terms.

The solution 7-15 fails if $D = 0$; the equations 7-13 are said to be **inconsistent**. If however all four of the determinants in 7-15 are zero, it is usually possible to assign a value to one of the unknowns at will, and then solve for the other two unknowns. In this case, the equations are **not independent**.

Example. Let us solve by means of determinants the equations

$$\begin{cases} 2x + 3y + z = 2 \\ 3x + y - 2z = 3 \\ 2y - 3z = 15 \end{cases}$$

The first step is to evaluate the determinant of the coefficients of the unknowns

$$D = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 0 & 2 & -3 \end{vmatrix} = \begin{matrix} 2(1)(-3) - 2(2)(-2) \\ + 3(2)(1) - 3(3)(-3) \\ + 0 \quad \quad \quad 0 \end{matrix}$$

$$= 35$$

Replace the elements in the first column by the constant terms, and evaluate the determinant

$$\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 15 & 2 & -3 \end{vmatrix} = -70$$

Replace the elements in the second column by the constant terms, and evaluate the determinant

$$\begin{vmatrix} 2 & 2 & 1 \\ 3 & 3 & -2 \\ 0 & 15 & -3 \end{vmatrix} = 105$$

Replace the elements in the third column by the constant terms, and evaluate the determinant

$$\begin{vmatrix} 2 & 3 & 2 \\ 3 & 1 & 3 \\ 0 & 2 & 15 \end{vmatrix} = -105$$

Hence

$$x = \frac{-70}{D} = -2$$

$$y = \frac{105}{D} = +3$$

$$z = \frac{-105}{D} = -3$$

Exercises

Evaluate the following determinants.

1. $\begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix}$

2. $\begin{vmatrix} 7 & 4 \\ 5 & -1 \end{vmatrix}$ *Ans.* -27

3. $\begin{vmatrix} a & c \\ a & d \end{vmatrix}$

4. $\begin{vmatrix} \sin A & -\cos A \\ \cos A & \sin A \end{vmatrix}$ *Ans.* 1

5. $\begin{vmatrix} 3 & 1 & -2 \\ 6 & 0 & 1 \\ 2 & -2 & 4 \end{vmatrix}$

6. $\begin{vmatrix} 5 & -1 & 7 \\ -4 & 1 & -2 \\ -3 & 0 & 6 \end{vmatrix}$ *Ans.* 21

Solve the following sets of simultaneous equations by means of determinants.

7. $\begin{cases} 3x - 7y = 12 \\ 2x + 4y = -5 \end{cases}$

8. $\begin{cases} 2x + 5y = -5 \\ 4x - 3y = 16 \end{cases}$ *Ans.* $\frac{5}{2}, -2$

9. $\begin{cases} x - 2y + z = 8 \\ 3x + 2y - z = -4 \\ 2x + 5y - 2z = -14 \end{cases}$

10. $\begin{cases} 4x - y + 2z = -1 \\ x + 2y - 3z = -1 \\ 3x - 4y + z = -8 \end{cases}$ *Ans.* $-\frac{1}{2}, 2, \frac{3}{2}$

44. Minors. If the row and the column containing any element of a determinant are stricken out, the resulting determinant is called the minor of that element.

Example. The minor of b_1 in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ is } \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

45. Additional Properties of Determinants. One reason for the importance of determinants arises from the fact that they make possible the

establishment of general theorems for systems of n equations in any number of unknowns. The discussion of determinants of higher order is beyond the scope of this book; but the following theorems apply to determinants of any finite order. They are readily verified for determinants of the second or third order (see the exercises below).

7-16. *If the rows and the columns are interchanged, the value of the determinant is not changed.* Any theorem proved for rows is therefore true for columns.

7-17. *If every element in the first row be multiplied by its minor, and the signs + and - be affixed alternately, the algebraic sum of the resulting products is equal in value to the original determinant.* This theorem tells us how to evaluate a determinant of the third order in terms of determinants of the second order (see Exercise 2 below). In general, it provides the means by which determinants of any order are evaluated.

7-18. *If any two rows are interchanged, the sign of the determinant is changed, but the numerical value is not changed.* Hence any row (or any column) may be used to expand a determinant in the manner suggested by theorem 7-17.

7-19. *If the elements of any row are each multiplied by the same constant, the value of the determinant is multiplied by this constant.* This theorem permits the removal of common factors from any row or column.

7-20. *If the elements of any row be added to (or subtracted from) the corresponding elements of any other row, the value of the determinant is unchanged.* It follows that if two rows (or two columns) are identical, the determinant vanishes.

7-21. *If the elements of any row be multiplied by a constant, and the result added to the corresponding elements of another row, the value of the determinant is unchanged.* This theorem is sometimes useful for reducing a determinant to simpler form (for example, one in which every element save one in a certain row is zero) before applying theorem 7-17.

Exercises

Show that:

$$1. \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$2. \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$3. \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$4. K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ Ka_2 & Kb_2 & Kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$5. \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + a_2 & b_1 + b_2 & c_1 + c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$6. \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + Ka_2 & b_1 + Kb_2 & c_1 + Kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

7. By means of theorem 7-21, reduce the determinant

$$\begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 4 \\ 4 & 2 & 0 \end{vmatrix}$$

to an equivalent form in which every element, except those in the principal diagonal, is zero. (The principal diagonal runs from the upper left-hand element to the lower right-hand element.)

$$Ans. \begin{vmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

8. Reduce the determinant

$$\begin{vmatrix} 3 & 1 & -5 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{vmatrix}$$

to an equivalent form in which every element, except those in the principal diagonal, is zero. As a check, evaluate your answer and the given determinant by using theorem 7-17.

CHAPTER 8

THE THEORY OF EXPONENTS

46. The Laws of Exponents. The history of human knowledge exhibits over and over the profound and far-reaching consequences that flow from simple ideas. The invention (by Descartes, about 1640) of the modern exponent notation is a case in point. Exponents were invented in order to promote conciseness in writing certain formulas. The notation is unspectacular. Its introduction into algebra is merely an incident in the endless struggle of the mathematician to encompass more meaning with fewer symbols. The inventor himself foresaw but an insignificant part of the marvelous developments that were to follow.

Descartes' idea was that a superscript number be used to indicate the number of times a factor is repeated; so that, for example, $(7)(7)(7)(7)$ would be written in the abbreviated notation 7^4 . The superscript number is called an *exponent*. It was originally defined to mean *the number of times that a quantity is to be taken as a factor*. Mathematicians observed that certain formal rules of operation upon these superscript numbers can be deduced directly from the definition. One such rule is given by the equation

$$(N^a)(N^b) = N^{a+b} \qquad 8-1$$

This rule of operation is often called the *first law of exponents*.

Consider the special case where $a = 3$ and $b = 5$. Then $N^3 = (N)(N)(N)$ and $N^5 = (N)(N)(N)(N)(N)$; so that

$$\begin{aligned} (N^3)(N^5) &= [(N)(N)(N)][(N)(N)(N)(N)(N)] \\ &= N^8. \end{aligned}$$

The proof is readily generalized.

Thus the rule of operation 8-1 is a rather obvious consequence of the definition of exponents. What is not obvious is the effect of ignoring the original definition, and *allowing the rule of operation itself to define the meaning of the superscript*. This is the point of view presently to be adopted.

Another rule of operation is expressed by the equation

$$\frac{N^a}{N^b} = N^{a-b} \qquad 8-2$$

As we shall presently see, this may be regarded as an extension of equation 8-1.

Yet another rule of operation is represented by the equation

$$(MN)^a = M^a N^a \quad 8-3$$

which is called the *second law of exponents*.

Consider the special case $a = 6$. We have

$$\begin{aligned} (MN)^6 &= (MN)(MN)(MN)(MN)(MN)(MN) \\ &= (MMMMMM)(NNNNNN) \\ &= M^6 N^6 \end{aligned}$$

The proof is easily made general.

An obvious extension of the second law of exponents is

$$\left(\frac{M}{N}\right)^a = \frac{M^a}{N^a} \quad 8-4$$

Another important rule of operation is the following:

$$(N^a)^b = N^{ab} \quad 8-5$$

This is the *third law of exponents*.

In proof, we observe that the left-hand member is equivalent to $[(N)(N)(N) \cdots]^b$, where there are a factors within the bracket. When the bracketed quantity is multiplied by itself b times, the factor N is repeated ab times in all.

The foregoing results may be summed up as follows: If N^a be defined as the product of a factors, each equal to N , then the rules of operation represented by equations 8-1 to 8-5, and any other rules derived from them, are logical consequences of the definition. It is evident that the definition has meaning only when the exponent is a positive whole number.

The quantity N is called the *base*. It is helpful to remember that the first law applies to the multiplication (or division) of factors where *the bases are alike*; the second law applies where *the exponents are alike*; and the third law applies to *the power of a power* of a quantity.

In order to avoid ambiguity, it has been agreed that *an exponent affects only the quantity to which it is attached*.

Examples.

- (a) $-16^2 = -256$, but $(-16)^2 = +256$
- (b) $(2a)^5 = 32a^5$, but $(2 + a)^5$ does not equal $(32 + a^5)$
- (c) $\frac{N^{x+7} + N^{x+2}}{N^{x+3}} = \frac{N^{x+2}(N^5 + 1)}{N^{x+2}(N)} = \frac{N^5 + 1}{N}$

Exercises

1. Evaluate: $(-3)^4$; -3^4 ; $(-a)^3$; $[(-a)^2]^3$; $[-(a^2)]^3$.
2. What is the value of $(-1)^k$ when k is even? What if k is odd? What is the value of $(-1)^{2k+1}$ if k is any integer?
3. Simplify $\left[\frac{x^{2a}}{x^{a+b}}\right]^{a+b} [(x^b)^b]$. Ans. x^{a^2}
4. Translate into English the rules of operation 8-1 to 8-5.
5. Simplify $\frac{3^{n+4} - 3^{n+1}}{(3)(3^{n-1})}$. Ans. 78
6. Simplify $\frac{(2x)^{k+1}}{2x^{k-1}}$.
7. Divide R^{n^2-1} by R^{n-1} . Ans. R^{n^2-n}
8. Simplify $\sqrt{\frac{(x^{n+1})^{n-3}}{(x^{n-1})^{n+1}}}$. Ans. $\frac{1}{x^{n+1}}$
9. Simplify $\left[\frac{x^{2a}(x^{-b})}{x^{a-2b}}\right]^{a-b}$. Ans. $x^{a^2-b^2}$

47. Meaning of a Zero Exponent. The exercises of the preceding section show the power of the rules of operation 8-1 to 8-5. By their aid, algebraic manipulations are easily performed that would be exceedingly tedious, if one were forced to work directly from the definition of an exponent. Now, the very form of these rules of operation raises the question, *Why must the exponents be positive integers?* One answer is, of course, that the original definition of an exponent is meaningless, unless the exponent is a positive integer. Another approach to the question, far more general, and of great practical utility, is possible. We may reverse our point of view, and inquire whether the meaning of an exponent can be reinterpreted in such a way that the rules of operation are valid when the exponents represent any of the numbers of algebra.

The definition of an exponent implied the formal rules of operation; let us see whether it is possible to begin with the rules of operation, and thence deduce the meaning of an exponent. Since we often write a quantity without any exponent, it is desirable that the absence of an exponent be logically part of the theory, just as, in the theory of factoring, the absence of a coefficient means that the coefficient is 1. Let us make the convention that the exponent 1 shall correspond to no exponent at all; that is,

$$x^1 = x$$

Moreover, let us assume that the first law of exponents holds for all the numbers of algebra. Then

$$x^2 = (x^1)(x^1) = (x)(x)$$

$$x^3 = (x^2)(x^1) = (x)(x)(x) \quad \text{etc.}$$

That is, the meaning of positive integral exponents can be deduced logically from the rule of operation 8-1. The remaining rules of operation are deduced as before.

Next, let $a = 0$ in equation 8-1. The formal result is

$$(N^0)(N^b) = N^b$$

This can be true in general only if

$$N^0 = 1 \qquad \qquad \qquad \mathbf{8-7}$$

The zero-th power of any quantity (other than zero) is equal to 1; that is the only meaning consistent with the first law.

The above reasoning holds for every value of N , other than zero. If $N = 0$, the proof fails, and there is no reason to suppose that equation 8-7 is valid. We do not attempt to assign a meaning (in this course) to the expression 0^0 .

48. Negative Exponents. Proceeding in the same way, let a be any positive integer, and b be the corresponding negative integer. Then equation 8-1 becomes

$$(N^a)(N^{-a}) = N^{a-a} = 1$$

Hence

$$N^{-a} = \frac{1}{N^a} \qquad \qquad \qquad \mathbf{8-8}$$

Thus we discover the meaning of a negative exponent: two quantities which differ only in the sign of the respective exponents are reciprocals of each other. Another convenient interpretation of equation 8-8 is the following: *any factor may be transferred from the numerator to the denominator of a fraction, or vice versa, if the sign of the exponent be changed.*

Because of equation 8-8, we may write

$$\frac{N^a}{N^b} = (N^a)(N^{-b})$$

so that equation 8-2 is seen to be equivalent to equation 8-1. There is no difficulty in verifying that the remaining rules of operation are consistent with the foregoing treatment of zero and negative exponents.

Example.

$$\frac{5a^{-2}}{4b^{-3}} = \frac{5b^3}{4a^2}, \quad \text{but } \frac{5+a^{-2}}{4+b^{-3}} \text{ does not equal } \frac{5+b^3}{4+a^2}$$

Exercises

1. Simplify (a) $\frac{a^0(R^{-1})}{R^{-4}}$; (b) $\frac{p^{-2}-4q^{-2}}{p^{-1}+2q^{-1}}$; (c) $\frac{mn}{m^{-2}+n^{-2}}$.

2. Show that $\left(\frac{N}{M}\right)^{-a} = \left(\frac{M}{N}\right)^a$.

3. Simplify $(pw^{-1})^{-2}(w^2p)^{-1}$.

Ans. $\frac{1}{p^3}$

4. Write in expanded form, without negative exponents:

$$\sum_{i=1}^4 C_i^{-1}$$

5. Write the following expressions as products:

$$\frac{17}{x^3}, \quad \frac{3x}{(a-x)}, \quad \frac{x+2}{x-2}$$

6. Expand $(2x - 3x^{-1})^2$.

7. Simplify $\frac{p^{-2}}{p^{-1}+2q^{-1}}$.

Ans. $\frac{q}{p(2p+q)}$

8. Remove the negative exponent, and write as a simple fraction

$$\frac{e^x + e^{-x}}{2}$$

9. Simplify $\frac{x^{-p}((x^{-2p} - 4y^{2p}))}{x^{-p} + 2y^q}$.

Ans. $\frac{1 - 2x^p y^q}{x^{2p}}$

10. Simplify $\sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}$.

Ans. $\frac{e^x + e^{-x}}{2}$

11. Simplify $\frac{M^{-a} - N^{-a}}{N^{-2a} - M^{-2a}} + N^a$.

Ans. $\frac{N^{2a}}{N^a + M^a}$

49. The Real Number System of Algebra. Conciseness and generality: these are the gods of the algebraist. The attempt to make the formulas of algebra as simple and as general as possible has resulted in the number system now to be described.

Beginning with the number 1 and the operation of counting (addition), we may generate the *natural numbers*, 1, 2, 3, \dots . If the operation of subtraction be introduced, other numbers are needed; otherwise the

operation symbolized by $a-b$ lacks a meaning when b is equal to, or greater than, a . Adding to the positive integers the numbers $0, -1, -2, -3, \dots$, we attain full generality for all formulas derived by the operations of addition and subtraction.

The operation of multiplication may be introduced without disturbing our number system; but the inverse operation of division is meaningless in an annoying proportion of instances, unless fractions are introduced into the number system. Even then there is one exceptional case; we reluctantly admit the exception by saying that division by zero is excluded. To divide a number N by a number M means to find another number x such that $N = Mx$. If $M = 0$, it is necessary to find a number x such that $(0)(x) = N$. Unless N is zero, there is no such number; if N is zero, there are infinitely many such numbers. For this reason, division by zero is prohibited.

The numbers discussed heretofore are called *rational*. A rational number is defined as one that may be written in the form p/q , where p and q are integers, either positive or negative, that have no common factor, other than 1. The possibility that q may be zero is excluded; but q may have the value 1, so that ordinary whole numbers are included in the set of rational numbers.

These numbers are sufficient for all algebraic purposes until the operation of extracting roots is introduced. Numbers such as $\sqrt{2}$, $\sqrt{7}$, $\frac{-1 + \sqrt{3}}{2}$, cannot be written in the form p/q ; they are called *irrational*.

It is easy to show that a number like $\sqrt{2}$ is not rational; for, if it be assumed that $\sqrt{2}$ is rational, this assumption is found to lead to a contradiction.

Suppose that we assume

$$\sqrt{2} = \frac{p}{q}$$

where the fraction has been reduced to lowest terms, so that p and q have no common factor except 1. In particular, p and q are assumed to be not both even. Then

$$2 = \frac{p^2}{q^2}$$

$$\therefore p^2 = 2q^2$$

The number $2q^2$ is obviously divisible by 2. Hence p^2 is an even number. It follows that p must be even, since the square of any odd number is odd.

But if p is divisible by 2, p^2 must be divisible by 4. Then $2q^2$ must be divisible by 4, and q^2 must be divisible by 2. But this would mean that q could not be odd, and hence q , like p , is divisible by 2.

But by hypothesis p and q have no common factor other than 1. Hence the assumption that $\sqrt{2}$ can be represented in the form p/q must be false, because it leads to a contradiction.

It will be shown in Chapter 15 that *every repeating decimal can be expressed as a rational number*. Thus $1.181818 \dots = \frac{11}{11}$. It follows that irrational numbers when expressed in decimal form always lead to non-repeating, non-terminating decimals. When the value of an irrational number is expressed in decimal form, the latter is always an approximation.

A clever device has been invented by mathematicians for coming to grips with elusive objects of this kind. It operates by methodically squeezing the unknown object between two known ones; just as a baseball player, caught "in a pickle" while attempting to steal a base, is put out. We can say that $\sqrt{2}$ lies between 1.41 and 1.42; squeezing a bit more, we can say that it lies between 1.414 and 1.415. *It is possible to find two rational numbers, differing by less than any preassigned quantity, however small, and yet such that the elusive irrational is trapped between them.* Thus if one were challenged to produce two rational numbers which differ by less than one part in ten million, and such that one is greater than $\sqrt{2}$, and the other smaller, the response might be

$$1.414\ 213\ 5 < \sqrt{2} < 1.414\ 213\ 6$$

The irrational numbers mentioned above are all included in the class of numbers that are roots of algebraic equations. For example, $\sqrt{2}$ is a root of the equation

$$x^2 - 2 = 0$$

There are other irrational numbers that are not among the roots of any polynomial equation of finite degree. They are called *transcendental* numbers. The two most important transcendental numbers are $\pi = 3.14159 \dots$ and $e = 2.71828 \dots$.

All of these different kinds of numbers make up the *real numbers* of algebra. Any real number is either rational or irrational. If irrational, it may be either algebraic or transcendental; if rational, it may be an integer or a fraction. The real numbers may be visualized as lying along

a straight line which extends indefinitely in either direction (Figure 34). Choose a point, which is to correspond to the number 1. Next choose a unit of length, corresponding to the interval between successive integers. We now have a *scale*, along which the positive and negative integers, and zero, may be laid off. The location of any rational number may be found

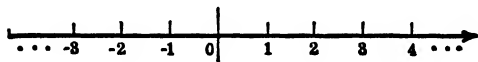


FIG. 34

by a simple geometrical construction. Between any two points on the scale, an infinite number of rational numbers may be located. Yet, crowded as the rational numbers appear to be, there are spaces among them, into which the irrational numbers are squeezed.

50. Fractional Exponents. The logical structure of the theory of exponents follows the pattern of the number system. We have seen how the first law of exponents can be used to attach a meaning to the exponent zero, and to positive and negative integral exponents. What if the exponent is a fraction?

Let us first consider the special case where the exponent is $\frac{1}{2}$. Then from equation 8-1 we have

$$(N^{\frac{1}{2}})(N^{\frac{1}{2}}) = N$$

$$N^{\frac{1}{2}} = \sqrt{N}$$

since it is one of two equal factors of N . In the general case,

$$(N^{\frac{p}{q}})(N^{\frac{p}{q}})(N^{\frac{p}{q}}) \cdots [q \text{ factors}] = N^p$$

Here $N^{\frac{p}{q}}$ is one of q equal factors of N^p . Hence

$$N^{\frac{p}{q}} = \sqrt[q]{N^p} \tag{8-9}$$

The denominator of the fraction is the index of the root. It is easily seen that this meaning is consistent with the other rules of operation.

This discovery makes obsolete the older *radical* notation for roots. Everything that can be done with the old notation can be done with the new; but the use of fractional exponents brings to our aid the powerful rules of operation 8-1 to 8-5. Though the use of radicals is still common, the trend among technical men is to favor the use of fractional (or decimal) exponents.

Example 1. The rules for operation with radicals depend logically upon the laws of exponents. To remove a factor from a radical, we may proceed thus:

$$\begin{aligned}\sqrt[a]{B^{3a+5}} &= B^{\frac{3a+5}{a}} \\ &= B^{3+\frac{5}{a}} \\ &= B^3(B^{\frac{5}{a}}) \\ &= B^3\sqrt[a]{B^5}\end{aligned}$$

To introduce a factor under the radical sign, the foregoing steps may be reversed.

Example 2. It is required to write under a single radical the expression $\sqrt{A}(\sqrt[3]{B})$.

Changing to fractional exponents, we have

$$\begin{aligned}A^{\frac{1}{2}}B^{\frac{1}{3}} &= A^{\frac{1}{2}}B^{\frac{1}{3}} \\ &= [A^{\frac{2}{3}}B^{\frac{2}{3}}]^{\frac{1}{2}} \\ &= \sqrt[6]{A^2B^2}\end{aligned}$$

Example 3. An expression containing radicals is in "simplest radical form" if (a) the least possible number of radicals is used; (b) no factor can be removed from a radical; (c) no index can be lowered; (d) there are no radicals in the denominator of a fraction; and (e) there are no fractions under the radical sign. The quotation marks are a reminder that this particular form is conventional.

For example, the expression $\frac{4\sqrt{a}}{\sqrt[3]{b}}$ is, practically speaking, in the simplest possible form. When reduced to the conventional "simplest radical form," it becomes $\frac{4}{b}\sqrt[6]{a^2b^4}$.

Let us reduce the expression

$$\left[\frac{\sqrt{m} + \sqrt{m-2}}{\sqrt{m} - \sqrt{m-2}} \right]^{\frac{1}{2}}$$

to the conventional form. The denominator of the fraction can be *rationalized* by multiplying by $(\sqrt{m} + \sqrt{m-2})$. In order not to change the value of the expression, the numerator of the fraction must be multiplied by the same quan-

tity. Hence we have

$$\begin{aligned} & \left[\frac{(\sqrt{m} + \sqrt{m-2})(\sqrt{m} + \sqrt{m-2})}{(\sqrt{m} - \sqrt{m-2})(\sqrt{m} + \sqrt{m-2})} \right]^{\frac{1}{2}} \\ &= \left[\frac{m + m - 2 + 2\sqrt{m(m-2)}}{m - (m-2)} \right]^{\frac{1}{2}} \\ &= \sqrt{m-1 + \sqrt{m(m-2)}} \end{aligned}$$

It is bad form to employ both radicals and fractional (or negative) exponents in expressing a result.

Example 4. Solve the equation

$$\sqrt{x-3} - \sqrt{x+4} = 1$$

In order to remove the radicals, it is necessary to square both members of the equation. Let us first rearrange the terms as follows:

$$\sqrt{x-3} = 1 + \sqrt{x+4}$$

Squaring both members, and equating the results,

$$x-3 = 1 + (x+4) + 2\sqrt{x+4}$$

Simplifying and rearranging the terms, we have

$$\sqrt{x+4} = -4$$

Squaring,

$$x+4 = 16$$

$$\therefore x = 12$$

Upon attempting to check this value, it is found not to satisfy the original equation. (It is conventional to consider that the radical sign indicates the *principal* root. This question is discussed in Chapter 13.) In fact, the given equation *has no root*. The value $x = 12$ is called an extraneous root; it was introduced by the process of squaring.

Exercises

1. Simplify $\left(\frac{x^{1.22}}{t^{0.77}}\right)^{3.1}$

Ans. $\frac{x^{3.8}}{t^{2.4}}$

2. What law of exponents was employed in Example 2 on page 84?

3. Multiply $(4p^{-\frac{1}{2}} - 2p^{-\frac{1}{2}} + 1)$ by $(2p^{-\frac{1}{2}} + 1)$.

Ans. $1 + \frac{8}{p}$

4. Multiply $(x^{-\frac{1}{2}} + 3x^{-\frac{1}{2}} + 9)$ by $(x^{-\frac{1}{2}} - 3)$.

5. Express in simplest radical form $\sqrt[3]{p\sqrt{a^9p}}$.

Ans. $\sqrt[3]{p}$

6. Express in simplest radical form $\sqrt{(3a)^{10}b^3\sqrt{b^2}}$.

7. Express in simplest radical form $\frac{\sqrt{a+b}}{\sqrt{a-b}} - \frac{\sqrt{a-b}}{\sqrt{a+b}}$.

8. Express in simplest radical form $\frac{\sqrt{a-b} - \sqrt{a+b}}{\sqrt{a-b} + \sqrt{a+b}}$. Ans. $\frac{\sqrt{a^2 - b^2} - a}{b}$

9. Solve for p :

$$\sqrt{2p} - \sqrt{2p-15} = 1$$

10. Solve for x :

$$\sqrt{9x} + \sqrt{9x+8} = 4$$

11. Simplify $3\sqrt{2 + \frac{b}{a}} - \sqrt{8a^2 + 4ab}$.

$$\text{Ans. } \left(\frac{3}{a} - 2\right)\sqrt{2a^2 + ab}$$

12. Simplify $(\sqrt[3]{\sqrt[4]{a^9}})^4$.

13. Multiply $\frac{a^{-2}b^5}{8a^{3/2}b^{-4}}$ by $\frac{b\sqrt{b}}{4a^4}$. Express your answer in simplest radical form.

$$\text{Ans. } \frac{b^{10}\sqrt{ab}}{32a^9}$$

14. Simplify $\sqrt{\frac{5-x^2}{2}} - \sqrt{90a^2 - 18a^2x^2}$.

15. Solve the equation

$$\frac{p}{p_0} = \left[1 - \frac{k-1}{2}\left(\frac{v}{v_0}\right)^2\right]^{\frac{k}{k-1}}$$

for v in terms of the other quantities.

$$\text{Ans. } \pm v_0 \sqrt{\frac{2}{k-1} \left[1 - \left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}\right]}$$

16. Simplify:

$$(a) \sqrt[3]{\frac{M}{6}} + \sqrt[3]{\frac{9}{2M^2}}$$

$$(b) \sqrt[3]{\frac{M}{6}} \left(\sqrt[3]{\frac{9}{2M^2}}\right)$$

$$(c) \sqrt[3]{\frac{M}{6}} \left(\sqrt[3]{\frac{9}{2M^2}}\right)$$

17. The pressure drop p of steam flowing through a pipe can be found from the formula

$$p = \left[\frac{0.512W^{0.85}y^{-2}L^3}{W^{-1}yd^{4.91}} \right]^{\frac{1}{3}}$$

where d is the diameter, and L the length, of the pipe. Simplify the expression.

$$\text{Ans. } \frac{0.8LW^{1.55}}{yd^{4.97}}$$

18. Solve the simultaneous equations

$$\begin{cases} x^3 + y^3 + z^3 = 9 \\ 2x^3 + 3z^3 = 16 \\ \frac{1}{2}x^3 + \frac{1}{3}y^3 - z^3 = -2 \end{cases}$$

19. Simplify

$$\frac{x^2(a^2 + x^2)^{-\frac{1}{2}} - (a^2 + x^2)^{\frac{1}{2}}}{x^2}$$

Ans. $\frac{a^2\sqrt{a^2 + x^2}}{x^2(a^2 + x^2)}$

20. Simplify

$$x^3(a^3 + x^3)^{-\frac{1}{3}} - (a^3 + x^3)^{\frac{1}{3}}$$

21. Divide $\frac{3x^{\frac{1}{2}}y^{-\frac{1}{2}} - 2x^{\frac{1}{2}}y^{-\frac{3}{2}}}{\sqrt{4xy^3}}$ by $\frac{9x^{\frac{3}{2}}y - \frac{4}{y}}{3x^{\frac{3}{2}}y + 2x}$.

Ans. $\frac{x}{2y^2}$

CHAPTER 9

LOGARITHMS

51. The Scientific Notation for Numbers. Consider the following table of powers of 10:

...	
$10^4 = 10,000$	
$10^3 = 1000$	
$10^2 = 100$	
$10^1 = 10$	9-1
$10^0 = 1$	
$10^{-1} = 0.1$	
$10^{-2} = 0.01$	
$10^{-3} = 0.001$	
...	

Numbers that are very large, or very small, may be concisely written by using powers of 10. For example, the charge on an electron is $4.80(10^{-10})$, instead of 0.000 000 000 480, electrostatic units. Avogadro's number is $6.06(10^{23})$, instead of 606,000,000,000,000,000,000,000.

Example. It is required to evaluate

$$\sqrt[3]{0.007\ 2^2 + 0.000\ 322}$$

Using powers of 10, we may proceed thus:

$$\begin{aligned} & \sqrt[3]{[7.2(10^{-3})]^2 + 3.22(10^{-4})} \\ &= \sqrt[3]{51.8(10^{-6}) + 3.22(10^{-4})} \\ &= \sqrt[3]{51.8(10^{-6}) + 322(10^{-6})} \\ &= \sqrt[3]{374(10^{-6})} \\ &= 7.20(10^{-2}) \end{aligned}$$

Another advantage of the scientific notation was mentioned in Chapter 2; the number of significant figures is always clearly indicated. Most of the quantities occurring in engineering and scientific work are like the irrational numbers of the mathematician; their *exact* values cannot be

written down, because they are not known. When they are expressed by means of rational numbers, the degree of approximation should be indicated.

A third advantage of the scientific notation is illustrated in the following example. Suppose that the expression

$$\frac{0.0137(0.000\ 375)^{\frac{1}{2}}}{(2360)^3}$$

is to be evaluated. The answer is 202 on the slide rule; but the position of the decimal point is not evident. Using the scientific notation, the problem may be written

$$\frac{1.37(10^{-2})[3.75(10^{-4})]^{\frac{1}{2}}}{[2.36(10^3)]^3} = \frac{1.37\sqrt{3.75}(10^{-2})(10^{-2})}{(2.36)^3(10^9)}$$

Making a rough mental calculation:

$$\frac{(1)(2)}{10} (10^{-2-2-9}) = 0.2(10^{-13})$$

Hence the answer is $0.202(10^{-13})$.

A number is expressed in standard form when the decimal point follows the first significant figure. In the example just discussed, the answer is

$$2.02(10^{-14})$$

when written in standard form.

Exercises

1. The average velocity of a molecule of argon gas, under standard conditions of temperature and pressure, is about 38,000 cm. per second. Write this number in standard form. Ans. $3.8(10^4)$
2. The conductivity of asbestos fiber is expressed by the coefficient $1.9(10^{-4})$. Write this number in the ordinary form.
3. Evaluate $\sqrt{a + b^{-1}}$ if $a = 2.71(10^{-5})$ and $b = 0.699(10^5)$.
4. Evaluate $\sqrt[3]{a^2 + b}$ if $a = 5.44(10^{-3})$ and $b = 56.8(10^{-6})$.
5. Find the value of $\frac{a}{\sqrt{b - c}}$ for $a = 426$, $b = 2.45(10^{-5})$, and $c = 0.0083(10^{-3})$. Ans. $1.06(10^5)$
6. Find the value of $\sqrt{x + \frac{y}{z}}$ if $x = 7,520,000$, $y = 43.1$, and $z = 0.000\ 086\ 4$. Ans. $2.83(10^3)$
7. Remove the factor 10^{-3} from the expression

$$15(10^{-4}) + 3.60(10^{-2}) + 0.801(10^{-1})$$

and simplify.

8. The critical potential of the hydrogen atom is given by

$$V = \frac{CHR}{E(10^8)} \left(1 - \frac{1}{n^2}\right)$$

Find V if $E = 1.592(10^{-20})$, $R = 109,700$, $C = 3.00(10^{10})$, $H = 6.54(10^{-27})$, and n is exactly 2. Ans. 10.1

52. The Laws of Logarithms. If the first law of exponents (equation 8-1) be written with the base $N = 10$ the result is

$$(10^x)(10^y) = 10^{x+y} \quad 9-2$$

If the exponent x is rational, 10^x represents some number A , on account of equation 8-9. If x is irrational, it can be expressed to any desired degree of approximation by rational numbers; hence 10^x can be expressed, at least in theory, to any desired degree of approximation by equation 8-9. In practice, the number A is approximated by means of a table of logarithms, as will be explained. The table itself is computed by methods lying outside the scope of this book.

We may set, then,

$$10^x = A \quad 9-3$$

By definition, x is the logarithm of A to the base 10. This is written

$$x = \log A \quad 9-4$$

In like manner, we may set

$$10^y = B$$

which is equivalent to

$$y = \log B$$

Now from equation 9-2, we have

$$AB = 10^{x+y}$$

$$\therefore \log (AB) = x + y$$

by definition. Thus we see that

$$\log (AB) = \log A + \log B \quad 9-5$$

This equation is essentially equivalent to the first law of exponents. In like manner, we find

$$\log \left(\frac{A}{B}\right) = \log A - \log B \quad 9-6$$

Moreover, it is easy to show that

$$\log (A^b) = b \log A \quad 9-7$$

This equation is equivalent to the third law of exponents

If the letter A be eliminated between equations 9-3 and 9-4, we obtain the formula

$$\log 10^x = x \quad 9-8$$

If, instead of A , the letter x is eliminated, we have

$$10^{\log A} = A \quad 9-9$$

The last two equations bring out clearly the fact that *the operation of taking a logarithm to the base 10 is inverse to the operation of raising 10 to a power*, in exactly the same sense that the operation of division is inverse to the operation of multiplication.

Exercises

1. By direct use of equation 9-3, find the numbers whose logarithms are: $\frac{1}{2}$; -3 ; $\frac{1}{3}$.

$$\text{Ans. } \sqrt{10}, 0.001, \sqrt[3]{10}$$

2. Can a logarithm be negative?

3. Using the laws of logarithms, express

$$2 \log 3 - \log 7 + \log 14 - 4 \log 2$$

as a single logarithm.

$$\text{Ans. } \log 1.125$$

4. Express the meaning of equations 9-5 to 9-9 in words.

5. From the table of powers of 10 at the beginning of this chapter, find $\log 0.001$; $\log 100$.

$$\text{Ans. } -3, 2.$$

6. Find $\log \sqrt[3]{1000}$.

7. Solve for x :

$$\log (2x - 148) = 2$$

$$\text{Ans. } 124$$

8. Solve for x :

$$\log (x^2 - 1) - \log (x + 1) = \frac{1}{2}$$

9. Express as a single logarithm, in simplest form:

$$2 \log (x + y) - \log (x^2 - y^2)$$

$$\text{Ans. } \log \frac{x + y}{x - y}$$

10. Simplify the expression

$$\log \sqrt{A^2 - B^2} - \frac{1}{2} \log (A - B)$$

11. Just two of the following equations are correct. Write the left-hand members of the others in simplest form.

$$(a) \frac{\log 100}{\log \sqrt{10}} = \frac{.3}{2}$$

$$(b) \log \left(\frac{A^2 B}{2A} \right) = \frac{B}{2}$$

$$(c) \log (x + A) - \log A = \log x$$

$$(d) 10^{3 \log \sqrt{x}} = x^{\frac{3}{2}}$$

$$(e) \log \log (10^a) = 0$$

$$(f) \log R - \log S = \frac{\log R}{\log S}$$

$$(g) \log \frac{1}{N} = -\log N$$

$$(h) \log \left(\frac{10a - 10x}{x - a} \right) = -1$$

$$(i) \log A - \log \frac{A}{A-1} = A - 1$$

$$(j) \log \frac{x}{2} = \frac{1}{2} \log x$$

$$(k) 2 \log (x - y) + \frac{1}{2} \log (x + y) = \frac{5}{2} \log (x^2 - y^2)$$

$$(l) \frac{1}{2} \log (x^2 + 4) = \pm \log (x + 2)$$

12. Simplify

$$\log \sqrt{\frac{1}{1+x} - \frac{1}{x} + \frac{1}{x^2}}$$

$$Ans. \quad -\frac{1}{2} \log x^2(1+x)$$

53. Natural Logarithms. In theory, almost any number can be used as the base of a system of logarithms; but there are just two bases of practical importance. *Common logarithms* are calculated to the base 10; in this book, the symbol *log* will refer only to the common logarithm. *Natural logarithms* are calculated to the transcendental base $e = 2.71828 \dots$. We shall use the symbol *ln* for natural logarithms. By definition, if

$$e^x = A \quad 9-10$$

then

$$x = \ln A \quad 9-11$$

By taking the common logarithm of both members of equation 9-10, we arrive at the useful *formula for change of base*.

$$x \log e = \log A$$

$$\ln A = \frac{\log A}{\log e} = \frac{\log A}{0.4343 \dots} \quad 9-12$$

54. Mantissa and Characteristic. It is convenient at times to consider that a logarithm consists of two parts: an integral part, called the *characteristic*; and a decimal part, called the *mantissa*. Referring once more to table 9-1, it can be seen that *for numbers lying between 1 and 10, the logarithms lie between 0 and 1*. These are given in decimal form in tables

of common logarithms; a decimal point is understood before the first digit of the logarithm.

The great advantage of the base 10 lies in the fact that if a table be constructed for the logarithms of numbers between 1 and 10, the logarithm of any number may be found by a simple mental operation; for if two numbers are alike except for the position of the decimal point, their logarithms differ by a whole number. Thus we find in a table of logarithms that $\log 2 = 0.30103$. Then

$$\begin{aligned}\log 20 &= \log (2)(10) = \log 2 + \log 10 \\ &= 0.30103 + 1 \\ &= 1.30103\end{aligned}$$

$$\begin{aligned}\log 200 &= \log (2)(10^2) = \log 2 + \log 10^2 \\ &= 2.30103\end{aligned}$$

$$\begin{aligned}\log 0.002 &= \log (2)(10^{-3}) = \log 2 + \log 10^{-3} \\ &= 0.30103 - 3 \\ &= -2.69897\end{aligned}$$

As a rough check, note from table 9-1 that $\log 20$ should lie between 1 and 2; $\log 200$ between 2 and 3; and $\log 0.002$ between -2 and -3 . It is sometimes preferable to express negative logarithms in a special double form; for example,

$$\log 0.002 = 7.30103 - 10$$

To find the logarithm of a number, then, one looks up the *mantissa* from a table of logarithms, and adds (with regard to sign) the *characteristic*, which is the exponent to which 10 is raised, when the number is written in standard form.

The use of the terms "mantissa" and "characteristic" is convenient from the point of view of the computer, since two distinct operations are ordinarily employed in finding the logarithm of a number. There is, however, an unfortunate implication that all logarithms lead a double life, so to speak. Of course, any number can be broken up into two (or more) parts, in an infinite number of ways. It happens to be convenient to consider, for some purposes, that a logarithm can be broken up into an integral part, and a non-integral part less than 1. It is well to remember, however, that the mantissa and characteristic are merely parts; the logarithm itself is a number, obeying all of the familiar rules of arithmetic.

Example 1. Let $N = 732,500 = 7.325(10^5)$.

The mantissa is $\log 7.325 = 0.86\ 481$, and the characteristic is 5. Therefore $\log N = 5.86\ 481$.

Example 2. Let $N = 0.01\ 442 = 1.442(10^{-2})$.

The mantissa is $\log 1.442 = 0.15\ 897$, and the characteristic is -2 . Therefore $\log N = -1.84\ 103$; or, $\log N = 8.15\ 897 - 10$.

55. Interpolation. Consider the following table:

x	$f(x)$
3	4
4	9

What is the value of the function when $x = 3.6$? If we assume that $f(x)$ is a linear function, $f(3.6)$ will be $\frac{6}{10}$ of the way from 4 to 9, or

$$\begin{aligned} f(3.6) &= 4 + \frac{6}{10}(9 - 4) \\ &= 7 \end{aligned}$$

Now let a and $a + h$ be successive values of the argument, and let Δ be the difference in tabulated values of the function. Let m be some fractional part of the interval h . Then for a linear function

$$f(a + mh) = f(a) + m\Delta \qquad 9-13$$

This is the formula for linear interpolation. It is based on the assumption that *the increase in the function is proportional to the increase in the argument*. This is shown schematically in the following table:

x	$f(x)$	
a	$f(a)$	
$a + mh$	$f(a + mh)$	$\begin{array}{c} \uparrow \\ m\Delta \\ \downarrow \end{array}$
$a + h$	$f(a + h)$	$\begin{array}{c} \Delta \\ \downarrow \end{array}$

Example 1. Find $\log 5.4827$.

From the tables, $\Delta = 0.00\ 008$. Taking $m = 0.7$, we have

N	$\log N$	$\log N = 0.73\ 894 + 0.7(0.00\ 008)$
5.482	0.73\ 894	$= 0.73\ 894 + 0.00\ 006$
5.4827	?	$= 0.73\ 900$
5.483	0.73\ 902	

The sixth place of decimals is dropped, for two reasons. In the first place, the value of Δ is known to one significant figure only. In the second place, $\log N$ is not a linear function of N , although it is approximately linear over a short interval.

Example 2. Find N if $\log N = 0.72\ 038$.

The tabular difference $\Delta = 0.00\ 009$. We observe that $m\Delta = 0.00\ 006$.

N	$\log N$	Hence $m = \frac{\Delta}{\Delta} = 0.7$, and
5.252	0.72 032	$N = a + mh$
?	0.72 038	$= 5.2527$
5.253	0.72 041	

Example 3. Find $\cos 38^\circ 41.4'$.

Here Δ is negative, since the cosine decreases as the angle increases (in the first quadrant).

A	$\cos A$	
$38^\circ 41'$	0.78 061	
$38^\circ 41.4'$?	Taking $m = 0.4$ and $\Delta = -0.00\ 018$,
$38^\circ 42'$	0.78 043	we have

$$\begin{aligned}\cos 38^\circ 41.4' &= 0.78\ 061 - 0.4(0.00\ 018) \\ &= 0.78\ 054\end{aligned}$$

56. Computation by Logarithms. The following examples illustrate the use of logarithms in solving various kinds of problems.

Example 1. Find $\sqrt[3]{-0.01\ 379}$.

The minus sign presents a difficulty, since negative numbers do not have real logarithms; but we see that the answer will be negative, and equal numerically to $\sqrt[3]{0.01\ 379}$. Let us set $x = (0.01\ 379)^{1/3}$. Then

$$\begin{aligned}\log x &= \frac{1}{3} \log 0.01\ 379 \\ &= \frac{1}{3} (8.13\ 956 - 10) \\ &= \frac{1}{3} (28.13\ 956 - 30) \\ &= 9.37\ 985 - 10 \\ x &= 0.2398\end{aligned}$$

$$\therefore \sqrt[3]{-0.01\ 379} = -0.2398$$

Example 2. Calculate the value of $0.675^{1.41}$.

Slide rule accuracy is sufficient. Let us set $x = 0.675^{1.41}$. Then

$$\begin{aligned}\log x &= 1.41 \log 0.675 \\ &= 1.41(9.829 - 10) \\ &= 1.41(-0.171) \\ &= -0.241 \\ &= 9.759 - 10 \\ \therefore x &= 0.574\end{aligned}$$

BB' , and $B'D$. We may take the base line as giving a fixed direction, and find the components of the vectors parallel and perpendicular to it.

Solution

Computation

For vector BB' , the horizontal component is

$$\begin{aligned} & 305.4 \cos 74^\circ 37' \\ & = 81.0 \text{ feet} \end{aligned}$$

$$\begin{aligned} \log 305.4 &= 2.48 \ 487 \\ \log \cos 74^\circ 37' &= 9.42 \ 370 - 10 \\ \hline &1.90 \ 857 \end{aligned}$$

The vertical component is

$$\begin{aligned} & 305.4 \sin 74^\circ 37' \\ & = 294.5 \text{ feet} \end{aligned}$$

$$\begin{aligned} \log 305.4 &= 2.48 \ 487 \\ \log \sin 74^\circ 37' &= 9.98 \ 415 - 10 \\ \hline &2.46 \ 902 \end{aligned}$$

For vector $B'D$, the horizontal component is

$$\begin{aligned} & 250.0 \sin 74^\circ 37' \\ & = 241.0 \text{ feet} \end{aligned}$$

$$\begin{aligned} \log 250.0 &= 2.39 \ 794 \\ \log \sin 74^\circ 37' &= 9.98 \ 415 - 10 \\ \hline &2.38 \ 209 \end{aligned}$$

The vertical component is

$$\begin{aligned} & 250.0 \cos 74^\circ 37' \\ & = 66.3 \text{ feet} \end{aligned}$$

$$\begin{aligned} \log 250.0 &= 2.39 \ 794 \\ \log \cos 74^\circ 37' &= 9.42 \ 370 - 10 \\ \hline &1.82 \ 164 \end{aligned}$$

For vector AD , the horizontal component is the (algebraic) sum of the horizontal components of AB , BB' , and $B'D$:

$$367.1 \text{ feet}$$

$$\begin{array}{r} 207.1 \\ (-) \ 81.0 \\ \hline 126.1 \\ (+) \ 241.0 \\ \hline 367.1 \end{array}$$

The vertical component of AD is:

$$360.8 \text{ feet}$$

$$\begin{array}{r} 0.0 \\ (+) \ 294.5 \\ (+) \ 66.3 \\ \hline 360.8 \end{array}$$

$$\therefore \tan A = \frac{360.8}{367.1}$$

$$A = 44^\circ 30'$$

$$\begin{aligned} \log 360.8 &= 12.55 \ 727 - 10 \\ (-) \log 367.1 &= 2.56 \ 478 \\ \hline \log \tan A &= 9.99 \ 249 - 10 \end{aligned}$$

Exercises

1. (a) Outline the logarithmic work for computing the value of an expression of the form

$$\frac{AB^{\frac{1}{2}}}{C^{0.2}}$$

(b) Find the value of $\frac{2.74\sqrt[3]{0.0642}}{\sqrt{-0.193}}$.

Ans. -1.52

2. Find the value of

$$\sqrt[3]{\frac{-195.4\sqrt{38.63}}{4856}}$$

3. Find the value of $T_2 = T_1 \left(\frac{P_2}{P_1} \right)^{\frac{0.297}{1.297}}$ if $T_1 = 492^\circ$, and $P_2 = 3.75P_1$.

Ans. 666°

4. Find the value of $2.88^{5.34}$.

Ans. 285

5. Find the value of $0.288^{0.534}$.

Ans. 0.514

6. Find the value of $0.288^{5.34}$.

7. Station P in a survey is located 2177.4 feet to the east, and 4225.1 feet to the south, of a traverse point of the U. S. Coast and Geodetic Survey. Find the distance and bearing of Station P from the traverse point.

8. Station B in a survey is located 1722.0 feet to the west, and 2341.4 feet to the south, of a traverse point of the U. S. Coast and Geodetic Survey. Find the distance and bearing of Station B from the traverse point.

9. Solve for x :

$$1.9^{2x} = 3.1$$

10. Solve for R :

$$0.50^R = 0.10$$

Ans. 3.3

11. Evaluate

$$10^{0.301}(10^{0.477})$$

Ans. 6

12. Evaluate

$$y = 300(\sin 12^\circ 23')^{2.85}$$

Ans. 3.61

13. Evaluate

$$(0.0072)^{0.43}$$

Ans. 0.12

14. Find $\log \frac{1}{B}$ if $\log B = 8.576 - 10$.

Ans. 1.424

15. Evaluate $\sqrt[7]{0.48750 + \sin^2 A}$ for $A = 24^\circ 45.0'$.

Ans. 0.942 92

16. Find the value of $\sqrt[3]{1.5964 - \sin^2 B}$ for $B = 44^\circ 41' 40''$.

Ans. 1.0328

17. Solve for M :

$$\log 2^M + \log \frac{3}{2} = \log 24$$

18. Solve for x :

$$2.48^{2.15x} = 4.35^{1.02x+1.78}$$

Ans. 5.77

19. If a body initially at a temperature T_1 is surrounded by air at a temperature T_0 , the temperature T after t minutes is given by

$$T = T_0 + (T_1 - T_0)e^{-kt}$$

where k is a constant whose value is to be determined. In an experiment a body at 55.0° was left in air at 15.0° . After 11.0 minutes, the temperature of the body was found to be 25.0° . What is the value of k ? Calculate the temperature 9 minutes later (that is, when $t = 20.0$).

$$\text{Ans. } \begin{array}{l} k = 0.126 \\ T = 18.2^\circ \end{array}$$

20. Solve for N :

$$\log \log N = 1$$

$$\text{Ans. } 10^{10}$$

Miscellaneous Problems

1. Factor $B^{2m} - A^{2m}$.

2. Simplify $\frac{(kx)^0}{kx^0}$.

3. Express as a power of $a - 3x$:

$$\frac{1}{\sqrt[3]{(a - 3x)^2}}$$

4. Write in simplest radical form

$$\sqrt{2a(\sqrt[3]{3a})}$$

5. Simplify

$$\frac{(B^{a-1})^{a+1}}{B^{a^2}}$$

6. Simplify

$$\frac{(R^2 + 3R)(R^2 - 1)^{-\frac{1}{2}} - (R^2 - 1)^{\frac{1}{2}}}{(R + 3)^2}$$

7. Simplify

$$\sqrt[3]{\frac{B^{2a+1}}{B^{1-2a}}}$$

8. Solve for P :

$$\begin{cases} N_1 = \frac{1}{3}N_2 + 10 \\ N_2 + \frac{1}{4}N_1 = P + 60 \\ \frac{1}{5}N_1 = P + 30 \end{cases}$$

9. When two condensers are connected in series, the equations

$$\begin{cases} E_1C_1 = E_2C_2 = EC \\ E = E_1 + E_2 \end{cases}$$

must be satisfied. Eliminate the quantities E_1 and E_2 from these equations, and solve for C , the capacitance of the combination.

10. Simplify $\frac{2^{n+3} - 2(2^n)}{2(2^{n+3})}$.

11. Simplify the right-hand member of the equation

$$W = \frac{p_2v_2^k v_2^{1-k} - p_1v_1^k v_1^{1-k}}{1 - k}$$

12. Solve for
- x
- :

$$\frac{3^a(27)^{2a}}{9^{4a}} = 3^x$$

13. (a) Express in the scientific notation

$$\frac{1}{R_1}, \frac{1}{R_2}, \frac{1}{R_3}, \frac{1}{R_4}$$

where $R_1 = 1500$ ohms, $R_2 = 6000$ ohms, $R_3 = 25,000$ ohms, and $R_4 = 100,000$ ohms.

- (b) Hence calculate the total resistance of a parallel combination of these four resistances, from the formula:

$$\frac{1}{R} = \sum_{i=1}^4 \frac{1}{R_i}$$

14. Solve the simultaneous equations:

$$\begin{cases} x^{-1} + y^{-1} = \frac{8}{3} \\ x^{-2} - y^{-2} = \frac{4}{3} \end{cases}$$

(The second equation may be factored directly. Do not clear of negative exponents until values for x^{-1} and y^{-1} have been found.)

15. Solve for
- r
- in terms of the other letters:

$$t_2 = t_1 \left(\frac{1}{r} \right)^{g-1}$$

16. Evaluate the expression

$$H = \frac{1600(T_2^4 - T_1^4)}{10^{12}}$$

for $T_2 = 2000^\circ \text{K.}$ and $T_1 = 500^\circ \text{K.}$ (The law of Stefan-Boltzmann gives this expression for the radiant heat transmitted by a certain black body at T_2 degrees Kelvin to another at temperature T_1 .)

17. Show that the simultaneous equations

$$\begin{cases} \frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} \\ P_1 V_1^k = P_2 V_2^k \end{cases}$$

can be combined to yield the equation

$$T_2 = T_1 \left(\frac{P_2}{P_1} \right)^{\frac{k-1}{k}} \dots$$

(This result gives the temperature after an adiabatic expansion in terms of the expansion ratio, P_2/P_1 , the original temperature, and the constant k .)

18. In the design of a two-stage air compressor, there is encountered the equation

$$\frac{P_b^{-\frac{1}{n}}}{P_a^{-\frac{1}{n}}} = P_c^{-\frac{n-1}{n}} P_b^{-\frac{1-2n}{n}}$$

Solve for the receiver pressure, P_b .

19. Find the value of

$$(0.125)^{-3} + \frac{5}{2 + 2^{-1}}$$

20. Simplify

$$\left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}\right)^2$$

21. An overflow flood-gate is designed to carry off a maximum of 360 cubic feet of water per second. The formula for the discharge from a rectangular overflow channel is

$$q = 3.3bH^{\frac{3}{2}}$$

where $b = 12$ feet is the breadth of the channel, and H is the head of water. What is the maximum head allowed for?

22. Solve for v in terms of the other letters:

$$f_m = \frac{1}{2}f_t + \sqrt{\frac{1}{4}f_t^2 + v^2}$$

(This formula occurs in the theory of strength of materials. The quantity f_m is the maximum stress occurring in a beam where f_t is the unit horizontal tension and v is the unit shear.)

23. In pulley designs the crown C of a pulley is determined by the formula

$$C = \frac{f^3}{25}$$

Solve for f and simplify.

24. The thickness of a uniformly loaded rectangular plate of length a and breadth b is given by the equation

$$t = b \sqrt[4]{\frac{pk}{s \left[1 + \left(\frac{b}{a} \right)^2 \right]}}$$

Solve for a in terms of the other letters.

25. Remove the negative exponents and simplify:

$$\frac{q^{-2}p^{-2}\sqrt{pq+1}}{p^2q^{-4}(p^2-q^{-2})^{\frac{1}{2}}}$$

26. The unit stress of a thick-walled cylinder is given by Clavarino's equation:

$$p = \frac{W}{3} \left(\frac{r_1^2 + 4r_2^2}{r_2^2 - r_1^2} \right)$$

Design data give values for p , the internal unit pressure, W , and the inner radius r_1 . Find the outer radius r_2 in terms of the other letters.

27. Solve the equation

$$\frac{h^2}{\mu r} - 1 = \sqrt{1 + \frac{Eh^2}{\mu^2}} \cos \theta$$

for r in terms of the other letters. (This equation gives the path of any body moving under an inverse square law of attraction; for example, a planet or a comet about the sun.)

28. The discharge from a nozzle is given by

$$q = 29.83 D^2 \sqrt{\frac{P_1}{\left(\frac{1}{c}\right)^2 - \left(\frac{D}{d}\right)^4}}$$

where

q = discharge in gallons per minute

d = diameter of pipe or hose

D = diameter of nozzle outlet

P_1 = pressure in pounds per square inch

c = coefficient of discharge

Solve for d in terms of the other quantities.

29. The weight of one cubic foot of saturated steam depends upon the pressure in the boiler according to the formula

$$W = \frac{P^{0.941}}{330.36}$$

where P is the pressure in pounds per square inch. What is W if the pressure is 280 pounds per square inch?

30. The capacitance of a power line is given by the formula

$$x = 2\pi f \left(80 + 741 \log \frac{D}{r} \right) (10^{-9})$$

Taking $f = 60$ cycles, find x for a line consisting of two conductors of radius $r = 0.23$ inches, spaced at a distance $D = 48$ inches.

31. Find t from the equation

$$Q = Q_0 e^{-kt}$$

if $k = 0.6$, $e = 2.72$, and $Q/Q_0 = 0.05$. (This equation gives the quantity of starch remaining untransformed to glucose after a time t , in the presence of an excess of sulphuric acid.)

32. Solve the equation

$$0.006 \cdot 51^{-2n} = 3.78$$

33. Solve the equation

$$X^{\frac{b^2+1}{b}} = AX^b$$

34. From the equation

$$\ln i = -\frac{RT}{L} + \ln I$$

show that

$$i = I e^{-\frac{RT}{L}}$$

(This equation represents the decaying current in an inductance-resistance circuit such as the field circuit of a large generator.)

35. In the design of a superheater for a boiler, it is necessary to solve the following equation:

$$\frac{1}{(t - 378)^{0.16}} = (0.172)(0.4) + 0.294$$

Find the value of t .

36. An electrical engineer testing a new type of rectifier finds the wave form at an intermediate stage to be represented by the equation

$$i = 16.7 \sin^{1.41} \omega t$$

Calculate the current i when $\omega t = 45^\circ$.

37. Relays are used in communication and signal circuits. In all such circuits there is a time factor, determined by the equation

$$i = I + (I_o - I)e^{-\frac{R_t}{L}t}$$

where

i = instantaneous current through relay

I = final current through relay

I_o = original current through relay

If a circuit containing a relay which operates when the current reaches 0.09 amperes is closed (which implies that $I_o = 0$) on a 10-volt supply, how soon will the relay operate? The constants of the circuit are

R_t = relay resistance (60 ohms)
plus circuit line resistance (20 ohms)

L = 5 henries

I = $\frac{1}{2}$ amperes

38. The pressure drop p of steam flowing through a pipe can be found from the equation

$$p = \frac{0.80W^{1.85}L}{yD^{4.97}}$$

where D is the inside pipe diameter.

(a) Calculate the size of pipe for $p = 5.0$, $W = 9.5$, $L = 1000$, and $y = 0.2737$.

(b) Show that

$$\log D = 0.201 \left(\log \frac{0.80L}{py} + 1.85 \log W \right)$$

39. The diameter a of a circular plate fixed rigidly around its circumference and subjected to a central load Q is given by the relation

$$a = be^{\frac{t^2 S}{0.435 Q}}$$

where e is the base of natural logarithms. Solve this equation for the thickness t in terms of the other letters.

40. An empirical formula for the drop in temperature of a gas flowing through a flue is

$$\ln \ln \frac{T_1}{t_1} - \ln \ln \frac{T_2}{t_1} = ML$$

Show that

$$T_1 = t_1 \left(\frac{T_2}{t_1} \right)^{e^{ML}}$$

Also, calculate T_1 for $t_1 = 1000$ degrees absolute, $T_2 = 800$ degrees absolute, and $ML = 1.5$.

41. The bending moments at the supports of a uniformly loaded continuous beam are given by the equations

$$\begin{cases} 4M_2 + M_3 = -\frac{1}{3}WL^2 \\ 2M_2 + 4M_3 = -\frac{1}{3}WL^2 \end{cases}$$

Find M_2 and M , in terms of W , the load per linear foot of beam, and L , the length of the span.

42. Determine the arbitrary constants a , b , and c in the equation $S = at^2 + bt + c$, given that

$$S = 1 \quad \text{when} \quad t = 1$$

$$S = 3 \quad \text{when} \quad t = 2$$

$$S = 9 \quad \text{when} \quad t = 3$$

43. Eliminate t from the equations

$$\begin{cases} V = gt + V_o \\ S = \frac{1}{2}gt^2 + V_o t \end{cases}$$

and solve for V .

44. Ten tons of an ore that is 50% zinc are obtained by mixing ore that is 60% zinc with ore that is 35% zinc. How many tons of each ore must be used?
45. From the equations

$$\begin{cases} \frac{1}{f} = \frac{1}{p} + \frac{1}{p_1} \\ p + p_1 = L \\ p - p_1 = a \end{cases}$$

deduce the formula

$$f = \frac{L^2 - a^2}{4L}$$

46. Solve the following system of equations:

$$\begin{cases} A_x + B_x - C = 0 \\ A_y + B_y - 1.25 - 1.50 = 0 \\ 4B_y - 6A_y = 0 \\ 4B_x - 6A_x = 0 \\ 16C - 6(1.50) = 0 \end{cases}$$

47. A certain company has on hand four grades of sulphuric acid. The strength and impurities of the various grades are indicated by the following table:

	I	II	III	IV
Concentration	95%	62%	77%	54%
Lead	0.091	0.012	0.001	0.013
Arsenic	0.022	0.005	0.050	0.036

An order specifying acid of 69% strength, containing not more than 0.02% lead and 0.03% arsenic is to be filled. What amounts of each grade must be taken to meet exactly the specifications?

CHAPTER 10

ANALYTICAL TRIGONOMETRY

57. The Addition Formulas. The figure shows two vectors \overline{OP}_1 and \overline{OP}_2 . Let A be the angle of vector \overline{OP}_1 , laid off in the standard way; in like manner, let B be the angle of vector \overline{OP}_2 . Then the angle between the two vectors is $(A - B)$. Let $\overline{OQ} = \overline{OP}_1 \cos (A - B)$ be the projection of \overline{OP}_1 upon \overline{OP}_2 (see equation 1-1).

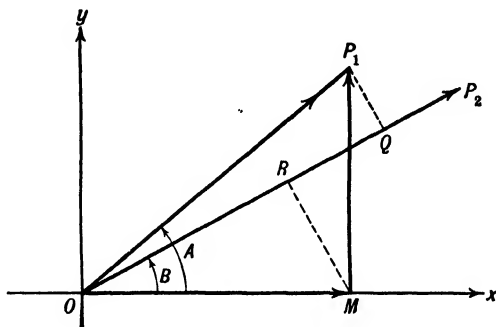


FIG. 36

The vector \overline{OP}_1 may be resolved into the components $\overline{OM} = \overline{OP}_1 \cos A$ and $\overline{MP}_1 = \overline{OP}_1 \sin A$. It is evident from Figure 36 that *the projection of \overline{OP}_1 upon \overline{OP}_2 is equal to the sum of the projections of the components of \overline{OP}_1 upon \overline{OP}_2 .* That is,

$$\overline{OQ} = \overline{OR} + \overline{RQ}$$

But

$$\overline{OR} = \overline{OM} \cos B = \overline{OP}_1 \cos A \cos B$$

And

$$\overline{RQ} = \overline{MP}_1 \sin B = \overline{OP}_1 \sin A \sin B$$

$$\therefore \overline{OP}_1 \cos (A - B) = \overline{OP}_1 \cos A \cos B + \overline{OP}_1 \sin A \sin B$$

Taking OP_1 to be a unit vector, we obtain the fundamental formula of analytical trigonometry:

$$\cos (A - B) = \cos A \cos B + \sin A \sin B \quad 10-1$$

As an indication of the generality of equation 10-1, we notice that, upon setting $A = B$, we have as a special case

$$1 = \cos^2 A + \sin^2 A$$

which is the Pythagorean relation 4-1.

A formula for the sine of the difference of two angles may be derived as follows:

$$\begin{aligned} \sin (A - B) &= \sqrt{1 - \cos^2 (A - B)} \\ &= \sqrt{1 - (\cos A \cos B + \sin A \sin B)^2} \\ &= \sqrt{1 - (\cos^2 A \cos^2 B + 2 \sin A \cos A \sin B \cos B + \sin^2 A \sin^2 B)} \end{aligned}$$

In the right-hand member, replace $\cos^2 A$ by $(1 - \sin^2 A)$, and $\sin^2 A$ by $(1 - \cos^2 A)$; noting that $(1 - \cos^2 B - \sin^2 B) = 0$, we have finally

$$\begin{aligned} \sin (A - B) &= \sqrt{\sin^2 A \cos^2 B - 2 \sin A \cos A \sin B \cos B + \cos^2 A \sin^2 B} \\ \therefore \sin (A - B) &= \sin A \cos B - \cos A \sin B \quad 10-2 \end{aligned}$$

If in equations 10-1 and 10-2 we set $A = 0$, we obtain

$$\cos (-B) = \cos B \quad 10-3$$

$$\sin (-B) = -\sin B \quad 10-4$$

Although, for convenience, the angles shown in Figure 36 are all less than 90° , the entire discussion remains valid if A and B take on any real values whatever, either positive or negative. Hence, we may replace B by $-B$ in equations 10-1 and 10-2; and with the aid of equations 10-3 and 10-4, it is easy to obtain the two *addition formulas*:

$$\cos (A + B) = \cos A \cos B - \sin A \sin B \quad 10-5$$

$$\sin (A + B) = \sin A \cos B + \cos A \sin B \quad 10-6$$

A formula for the tangent of the difference of two angles may be ob-

tained by dividing both members of equation 10-2 by the corresponding members of equation 10-1. The result is

$$\tan (A - B) = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

It is convenient to express this relation in terms of tangent functions only. If both numerator and denominator of the right-hand member be divided by $\cos A \cos B$, the result is

$$\begin{aligned} \tan (A - B) &= \frac{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}} \\ \therefore \tan (A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \end{aligned} \quad 10-7$$

In like manner, we find that

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad 10-8$$

58. Reduction to First Quadrant Angles. It is always possible to express the trigonometric functions of any real angle in terms of some function of a first quadrant angle. The complementary relations make possible a further reduction to angles of 45° or less.

An important special case is that of *supplementary angles*. Let us set $A = 180^\circ$ in equations 10-1, 10-2, and 10-7. The results are

$$\cos (180^\circ - B) = -\cos B \quad 10-9$$

$$\sin (180^\circ - B) = +\sin B \quad 10-10$$

$$\tan (180^\circ - B) = -\tan B \quad 10-11$$

Other special cases are left as exercises for the student. All of these will be found to confirm the rule:

Any trigonometric function of an even multiple of 90° plus or minus an angle is numerically equal to the same function of the angle; any function of an odd multiple of 90° plus or minus an angle is numerically equal to the co-named function of the angle. The sign is determined by the quadrant.

Example. Express $\tan 210^\circ$ in radical form.

$$\begin{aligned}\tan 210^\circ &= \tan (180^\circ + 30^\circ) \\ &= +\tan 30^\circ \\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

The $+$ sign is attached, since the tangent is positive in the third quadrant (page 44).

Exercises

1. Express in radical form, and check by tables:

(a) $\sin 15^\circ$

(c) $\sin 105^\circ$

(b) $\cos 75^\circ$

(d) $\tan 75^\circ$

2. Prove that $\tan (-B) = -\tan B$, by making use of equation 10-7.

3. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{1}{3}$ (A and B less than 90°), find

(a) $\sin (A + B)$

(b) $\cos (A - B)$

(c) $\tan (A + B)$

Ans. (a) $\frac{5}{13}$; (b) $\frac{4}{5}$; (c) $\frac{5}{13}$

4. If $\cos A = \frac{7}{5}$ and $\sin B = \frac{5}{13}$ (A and B less than 90°), find

(a) $\sin (A + B)$

(b) $\cos (A - B)$

(c) $\tan (A + B)$

5. Derive equation 10-2 by finding the projections of the vector \overline{OP}_1 (Figure 36) and its components in a direction *perpendicular* to the vector \overline{OP}_2 .

6. Prove the identities

(a) $\cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$

(b) $\cos \left(\frac{\pi}{4} + A \right) + \sin \left(\frac{\pi}{4} + A \right) = \sqrt{2} \cos A$

7. Show that

(a) $\sin (90^\circ + B) = \cos B$

(b) $\cos (90^\circ + B) = -\sin B$

(c) $\tan (90^\circ + B) = -\cot B$ [Suggestion: use equations 10-5 and 10-6].

8. Show that

(a) $\sin (\pi + B) = -\sin B$

(b) $\cos (\pi + B) = -\cos B$

(c) $\tan (\pi + B) = \tan B$

9. Reduce the expression

$$\cos B \cos (A + B) + \sin B \sin (A + B)$$

to simplest form.

Ans. $\cos A$

10. Reduce to simplest form

$$\cos B \cos (A - B) - \sin B \sin (A - B)$$

59. Functions of Twice an Angle. In equation 10-6 we may set $A = B$, thus obtaining the identity

$$\sin 2A = 2 \sin A \cos A \quad 10-12$$

In like manner, equation 10-5 yields the identity

$$\cos 2A = \cos^2 A - \sin^2 A \quad 10-13$$

The last result may be transformed by means of equation 4-1 into the useful alternative forms

$$\cos 2A = 1 - 2 \sin^2 A \quad 10-14$$

$$\cos 2A = 2 \cos^2 A - 1 \quad 10-15$$

60. Functions of Half of an Angle. It is permissible to replace A by $\frac{A}{2}$ in these identities; whence

$$\cos A = 1 - 2 \sin^2 \frac{A}{2}$$

Or

$$\sin^2 \frac{A}{2} = \frac{1}{2} - \frac{1}{2} \cos A \quad 10-16$$

From equation 10-15, we have

$$\cos A = 2 \cos^2 \frac{A}{2} - 1$$

Or

$$\cos^2 \frac{A}{2} = \frac{1}{2} + \frac{1}{2} \cos A \quad 10-17$$

Example. Express $\cos 105^\circ$ in radical form.

$$\begin{aligned} \cos 105^\circ &= \cos \frac{210^\circ}{2} \\ &= -\sqrt{\frac{1}{2} + \frac{1}{2} \cos 210^\circ} \\ &= -\sqrt{\frac{1}{2} + \frac{1}{2} \left(-\frac{\sqrt{3}}{2} \right)} \\ &= -\frac{1}{2} \sqrt{2 - \sqrt{3}} \end{aligned}$$

The negative sign must be chosen, since the cosine is negative in the second quadrant.

Another solution makes use of equation 10-5:

$$\begin{aligned}\cos 105^\circ &= \cos (60^\circ + 45^\circ) \\ &= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ \\ &= \frac{1}{2} \left(\frac{\sqrt{2}}{2} \right) - \frac{\sqrt{3}}{2} \left(\frac{\sqrt{2}}{2} \right) \\ &= \frac{\sqrt{2}(1 - \sqrt{3})}{4}\end{aligned}$$

That both answers are the same in value is easily shown by squaring both.

61. Some Applications. The formulas of analytical trigonometry afford a useful and flexible tool in the never-ceasing search for the simplest mathematical expression of physical laws.

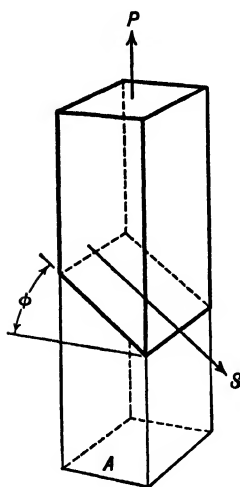


FIG. 37

Example 1. When a rod of cross-section A is subjected to a load P , the shearing stress per unit area produced in a section at an angle ϕ with the normal section will be

$$S = \frac{P \sin \phi}{A \sec \phi}$$

By equations 4-2 and 10-12, we have

$$\begin{aligned}S &= \frac{P}{A} \sin \phi \cos \phi \\ &= \frac{P}{2A} \sin 2\phi\end{aligned}$$

From this last equation, we may easily determine the angle at which fracture will occur, if the material is homogeneous; for the maximum value which $\sin 2\phi$ can take on is 1. Hence fracture occurs along a 45°

plane, and the maximum shearing stress is $\frac{P}{2A}$.

Example 2. A body of weight W , resting on an inclined plane, is acted upon by a force P as shown in Figure 38. It is required to obtain a formula for the force P which is just sufficient to overcome the friction F and the pull of gravity, so that the body is on the point of moving up the plane.

The normal reaction of the plane is

$$(W \cos \theta - P \sin \alpha)$$

The frictional force is found by multiplying the normal reaction by the coefficient of friction μ :

$$F = \mu(W \cos \theta - P \sin \alpha)$$

Summing up the forces acting along the plane, we see that

$$\begin{aligned} P \cos \alpha &= W \sin \theta + F \\ &= W \sin \theta + \mu W \cos \theta - \mu P \sin \alpha \end{aligned}$$

Solving for P , we find

$$P = W \frac{\sin \theta + \mu \cos \theta}{\cos \alpha + \mu \sin \alpha}$$

This formula can be simplified by writing

$$\mu = \tan \phi$$

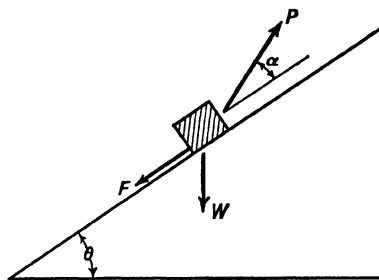


FIG. 38

The angle ϕ is called the angle of friction.

$$\begin{aligned} P &= W \frac{\sin \theta + \frac{\sin \phi \cos \theta}{\cos \phi}}{\cos \alpha + \frac{\sin \phi \sin \alpha}{\cos \phi}} \\ &= W \frac{\sin \theta \cos \phi + \cos \theta \sin \phi}{\cos \alpha \cos \phi + \sin \alpha \sin \phi} \\ &= W \frac{\sin (\theta + \phi)}{\cos (\alpha - \phi)} \end{aligned}$$

There is an evident gain in simplicity, that results from applying the formulas of analytical trigonometry. Suppose it is desired to know at what angle α the force P should be applied so that the body will move with the least effort. Evidently the denominator of the fraction should be as large as possible; but the largest possible value for the cosine is 1; it will take on this value when

$$\alpha - \phi = 0$$

$$\therefore \alpha = \phi$$

The force P will be a minimum when the angle α at which it is applied equals the angle of friction.

Additional illustrations of the use of analytical trigonometry are provided in the exercises below. See also Chapters 18 and 19.

Exercises

1. If $\tan A = \frac{5}{12}$ and A is less than 90° , find

- (a) $\sin 2A$
 (b) $\cos 2A$
 (c) $\tan 2A$
 (d) $\sin \frac{A}{2}$
 (e) $\cos \frac{A}{2}$

2. Establish the identities

$$(a) \cot B + \tan B = \frac{2}{\sin 2B}$$

$$(b) \frac{\sec 2x - 1}{\sec 2x + 1} = \tan^2 x$$

3. Express $\sin 3A$ in terms of $\sin A$.

$$Ans. \sin A(3 - 4 \sin^2 A)$$

4. Express $\cos 3A$ in terms of $\cos A$.

5. A body of weight W rests on an inclined plane, as in Figure 38. The force P which is just sufficient to prevent the body from sliding *down* the plane is given by the formula

$$P = W \frac{\sin \theta - \mu \cos \theta}{\cos \alpha - \mu \sin \alpha}$$

Set $\mu = \tan \phi$, and simplify by means of the multiple angle formulas.

$$Ans. W \frac{\sin(\theta - \phi)}{\cos(\alpha + \phi)}$$

6. The direction of the planes along which the principal stresses occur in a machine part can be found from the equation

$$S_s \frac{\cos^2 \theta}{\sin \theta} - S_t \cos \theta = S_s \sin \theta$$

where S_s is the unit shearing stress, and S_t the unit tensile stress. Find a simple expression for $\tan 2\theta$, from this equation.

$$Ans. \frac{2S_s}{S_t}$$

62. Other Formulas. The literature of applied mathematics contains a very large number of formulas connecting the trigonometric functions of one or more angles. Elaborate lists of these formulas may be found in reference books such as the *Smithsonian Mathematical Formulas and Tables of Elliptic Functions*. Of them all, the formulas printed in bold-face type in this chapter are most often used; they should be committed to memory, together with their meaning as expressed in words.

The other formulas that are numbered in this chapter are of secondary

importance; but the student of engineering should be aware that they exist, and should have a working knowledge of their use.

It is often convenient to express *the sum of the sines* of two angles in the form of a product. Adding the corresponding members of the equations

$$\sin (x+y)=\sin x \cos y+\cos x \sin y$$

$$\sin (x-y)=\sin x \cos y-\cos x \sin y$$

we obtain

$$\sin (x+y)+\sin (x-y)=2 \sin x \cos y$$

Now by setting $x+y=A$, and $x-y=B$ (whence $x=\frac{A+B}{2}$, and $y=\frac{A-B}{2}$), the result is

$$\sin A+\sin B=2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \quad 10-18$$

There are three other formulas of the same kind, obtained in a similar way. They are

$$\sin A-\sin B=2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \quad 10-19$$

$$\cos A+\cos B=2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \quad 10-20$$

$$\cos A-\cos B=-2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \quad 10-21$$

If it is desired to express *the product of two sine or cosine functions* in the form of a sum, formulas 10-18, 10-21, and 10-20 may be written thus:

$$\sin x \cos y=\frac{1}{2} \sin (x+y)+\frac{1}{2} \sin (x-y) \quad 10-22$$

$$\sin x \sin y=\frac{1}{2} \cos (x-y)-\frac{1}{2} \cos (x+y) \quad 10-23$$

$$\cos x \cos y=\frac{1}{2} \cos (x+y)+\frac{1}{2} \cos (x-y) \quad 10-24$$

It is possible to express *the sum of two sine or cosine terms with arbitrary coefficients* in the form of a single sine or cosine term. For example,

$$3 \sin x+4 \cos x=5\left(\frac{3}{5} \sin x+\frac{4}{5} \cos x\right)$$

By removing the factor $5(=\sqrt{3^2+4^2})$, we have obtained a form in which

the coefficient of $\sin x$ may be regarded as the sine of a particular angle ϕ , and the coefficient of $\cos x$ may be regarded as the cosine of the same angle ϕ . Hence

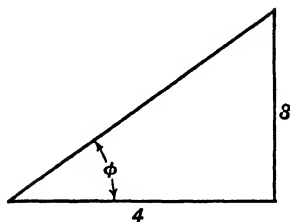


FIG. 39

$$\begin{aligned} 3 \sin x + 4 \cos x &= 5 (\sin \phi \sin x + \cos \phi \cos x) \\ &= 5 \cos (x - \phi) \end{aligned}$$

where the angle $\phi = 36^\circ 52'$ is the angle whose tangent is $\frac{3}{4}$ (see Figure 39).

In like manner we obtain the formulas

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin (x + \phi) \quad 10-25$$

Where

$$\tan \phi = \frac{b}{a}$$

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \cos (x - \phi) \quad 10-26$$

Where

$$\tan \phi = \frac{a}{b}$$

Exercises

1. Verify equations 10-22 to 10-26 by direct expansion of the right-hand members.
2. Verify equation 10-20 by direct expansion of the right-hand member.

Establish the following formulas by reducing both members to a common form:

$$3. \cos \frac{A}{2} \cos \frac{B}{2} - \sin \frac{A}{2} \sin \frac{B}{2} = \cos \frac{A+B}{2}$$

$$4. \sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$$

$$5. \cos A = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$$

$$6. \tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$$

$$7. \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}}$$

$$8. \tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A}$$

$$9. \tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}$$

$$10. \tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}$$

CHAPTER 11

OBLIQUE TRIANGLES

63. The Law of Sines. The solution of oblique triangles depends upon the properties of right triangles, since any triangle may be considered to be composed of right triangles. By carrying out the solution in literal terms, various useful formulas may be obtained.

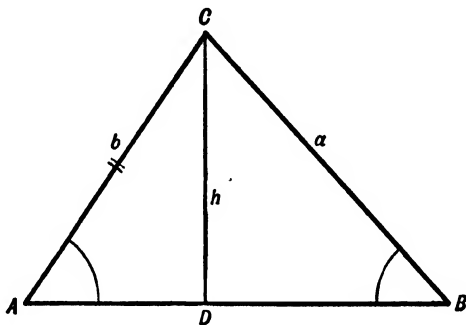


FIG. 40

In Figure 40, let us imagine that angles A and B are known, together with side b . We seek to express side a in terms of the known quantities. In the right triangle ACD , the hypotenuse and an acute angle are known. After the altitude h is found, the second right triangle BCD may be solved for a .

$$h = b \sin A$$

But from triangle BCD , we have

$$a = \frac{h}{\sin B} = \frac{b \sin A}{\sin B}$$

This result may be thrown into the symmetrical form

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Since the triangle is general, and the lettering arbitrary, we may inter-

change the letters B and C ; thus we arrive at the formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad 11-1$$

which is called the *law of sines*. The symmetry of the law of sines makes it easy to remember; and the fact that it is expressed in the form of a double proportion makes it admirably adapted to computation.

64. The Solution of Oblique Triangles, Cases I and II. The equations 11-1 express two independent conditions upon the six parts a, b, c, A, B, C . An additional condition is given by the relation

$$A + B + C = 180^\circ \quad 11-2$$

In theory, then, if any three independent parts (that is, other than the three angles) are known, the remaining three can be obtained by means of equations 11-1 and 11-2. In practice, the use of the law of sines is avoided unless both factors of one member are known. All problems in the solution of oblique triangles fall under one or other of the following four cases.

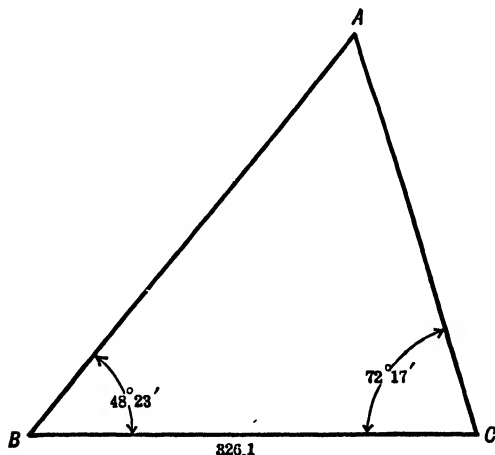


FIG. 41

Case I. *Given two angles, and any side.* Let $a = 326.1$, $B = 48^\circ 23'$, and $C = 72^\circ 17'$ (Figure 41). The third angle is calculated by equation 11-2, and the remaining parts are found by the law of sines.

Solution

$$\begin{aligned} A &= 180^\circ - (B + C) \\ &= 59^\circ 20' \text{ Ans.} \end{aligned}$$

$$b = \frac{a \sin B}{\sin A}$$

$$= 283.4 \text{ Ans.}$$

$$c = \frac{a \sin C}{\sin A}$$

$$361.1 \text{ Ans.}$$

Computation

$$\begin{aligned} 180^\circ &= 179^\circ 60' \\ B + C &= 120^\circ 40' \\ \hline A &= 59^\circ 20' \end{aligned}$$

$$\begin{aligned} \log 326.1 &= 2.51 \ 335 \\ \log \sin 59^\circ 20' &= 9.93 \ 457 - 10 \end{aligned}$$

$$\log \frac{a}{\sin A} = 2.57 \ 878$$

$$\begin{aligned} \log \sin 48^\circ 23' &= 9.87 \ 367 - 10 \\ \hline \log b &= 2.45 \ 245 \end{aligned}$$

$$\log \frac{a}{\sin A} = 2.57 \ 878$$

$$\begin{aligned} \log \sin 72^\circ 17' &= 9.97 \ 890 - 10 \\ \hline \log c &= 2.55 \ 768 \end{aligned}$$

A satisfactory check is afforded by the law of sines, using natural functions; the advantages in simplicity and directness outweigh the theoretical objection to employing the same formula for the check that is used for the solution. The student should review at this point the discussion of checking in Chapter 3. In using the law of sines for checking, it is necessary to verify that the sum of the three angles is 180° . The computations are facilitated by use of the contracted form of long division (see Chapter 2).

$$\begin{aligned} A &= 59^\circ 20' \\ B &= 48^\circ 23' \\ C &= 72^\circ 17' \end{aligned}$$

$$A + B + C = 179^\circ 60'$$

$$\begin{aligned} \frac{a}{\sin A} &= \frac{326.1}{0.860 \ 15} \\ &= 379.1 \end{aligned}$$

$$\begin{aligned} \frac{b}{\sin B} &= \frac{283.4}{0.747 \ 60} \\ &= 379.1 \end{aligned}$$

$$\begin{aligned} \frac{c}{\sin C} &= \frac{361.1}{0.952 \ 57} \\ &= 379.1 \end{aligned}$$

$$\begin{array}{r} 3261 \quad | 86015 \\ \hline 25804 \quad 3791 \\ \hline 6806 \end{array}$$

$$\begin{array}{r} 2834 \quad | 7476 \\ \hline 22428 \quad 3791 \\ \hline 5912 \end{array}$$

$$\begin{array}{r} 3611 \quad | 95257 \\ \hline 28577 \quad 3791 \\ \hline 7533 \end{array}$$

Case II. *Given two sides, and the angle opposite one of the known sides.* Let us find the smaller of the two triangles determined by the parts $a = 0.4162$, $b = 0.3677$, and $B = 54^\circ 18'$ (Figure 42). If the figure be con-

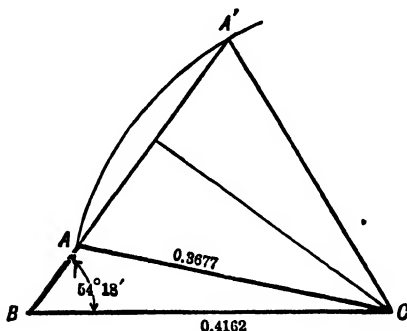


FIG. 42

structed to scale, it is easy to see that both triangle ABC and triangle $A'BC$ satisfy the given conditions. Since angle BAC and angle $BA'C$ are supplementary, $\sin A = \sin A'$.

Solution

$$\sin A = \frac{a \sin B}{b}$$

$$A = 113^\circ 11' \text{ Ans.}$$

$$C = 180 - (A + B)$$

$$= 12^\circ 31' \text{ Ans.}$$

$$c = \frac{b \sin C}{\sin B}$$

$$= 0.0981 \text{ Ans.}$$

Computation

$$\log 0.4162 = 9.61 \ 930 - 10$$

$$\log \sin 54^\circ 18' = 9.90 \ 960 - 10$$

$$\log (a \sin B) = 19.52 \ 890 - 20$$

$$\log 0.3677 = 9.56 \ 549 - 10$$

$$\log \sin A = 9.96 \ 341 - 10$$

$$180^\circ = 179^\circ 60'$$

$$A + B = 167^\circ 29'$$

$$C = 12^\circ 31'$$

$$\log 0.3677 = 9.56 \ 549 - 10$$

$$\log \sin 12^\circ 31' = 9.33 \ 591 - 10$$

$$\log (b \sin C) = 18.90 \ 140 - 10$$

$$\log \sin 54^\circ 18' = 9.90 \ 960 - 10$$

$$\log c = 8.99 \ 180 - 10$$

The law of sines, using natural functions, again provides a satisfactory check.

$$\frac{c}{\sin C} = \frac{0.0981}{0.2167}$$

$$= 0.453 \text{ (by slide rule).}$$

$$\frac{a}{\sin A} = \frac{0.4162}{0.919 \ 25}$$

$$= 0.4528$$

$$\frac{b}{\sin B} = \frac{0.3677}{0.812 \ 08}$$

$$= 0.4528$$

$$A = 113^{\circ}11'$$

$$B = 54^{\circ}18'$$

$$C = 12^{\circ}31'$$

$$A + B + C = 179^{\circ}60'$$

$$\begin{array}{r} 4162 \quad |9192\cancel{5} \\ 36770 \quad 4528 \\ \hline 4850 \end{array}$$

$$\begin{array}{r} 3677 \quad |8120\cancel{8} \\ 32483 \quad 4528 \\ \hline 4287 \end{array}$$

Case II is called the *ambiguous case*, because the given data may be satisfied by two triangles, by one triangle, or by no triangle. A figure that is *constructed*, to scale, and not merely sketched, will help to guard against errors in reasoning.

Exercises

It is recommended that the slide rule be used for all computations when the given data are expressed to three significant figures or less.

When the given data are expressed to four significant figures, it is recommended that five place tables be employed. Interpolation is not necessary.

When the given data are expressed to five significant figures, five place tables may be used, but interpolation is necessary.

- | | | |
|--|--|---|
| 1. $A = 66^{\circ}44'$
$b = 0.3127$
$C = 52^{\circ}59'$ | 2. $A = 76^{\circ}12'$
$B = 68^{\circ}17'$
$b = 1766$ | <i>Ans.</i> $a = 1846$
$c = 1104$
$C = 35^{\circ}31'$ |
| 3. $c = 512.14$
$B = 35^{\circ}17.8'$
$C = 46^{\circ}51.3'$ | 4. $A = 56^{\circ}44.5'$
$c = 42 \ 573$
$C = 72^{\circ}13.7'$ | <i>Ans.</i> $a = 37.382$
$b = 34.758$
$B = 51^{\circ}1.8'$ |
| 5. $b = 219.6$
$B = 34^{\circ}33'$
$c = 347.7$ | 6. $a = 6551$
$c = 5977$
$C = 62^{\circ}15'$ | <i>Ans.</i> $A = 75^{\circ}56'$ $A' = 104^{\circ}4'$
$b = 4503$ or $b' = 1597$
$B = 41^{\circ}49'$ $B' = 13^{\circ}41'$ |
| 7. $b = 74.530$
$B = 61^{\circ}19.2'$
$c = 82.166$ | 8. $a = 143.99$
$A = 103^{\circ}14.4'$
$b = 126.44$ | <i>Ans.</i> $B = 58^{\circ}44.2'$
$c = 45.77$
$C = 18^{\circ}1.4'$ |
| 9. $a = 317$
$c = 288$
$C = 41^{\circ}30'$ | 10. $b = 0.139$
$B = 101^{\circ}15'$
$c = 0.106$ | <i>Ans.</i> $a = 0.0712$
$A = 30^{\circ}15'$
$C = 48^{\circ}30'$ |

11. To find the distance from station P on one side of a river to station Q on the other side, a third station M is located on the same side as station P . The distance MP is found to be 385 feet. Angle PMQ is found to be $33^\circ 20'$, and angle MPQ is $117^\circ 35'$. Find PQ . Ans. 435 feet
12. A problem in mechanics leads to the equation

$$\frac{2000}{\sin 148^\circ 20'} = \frac{R_2}{\sin 163^\circ 20'}$$

Find the reaction R_2 .

13. From a boat the angles of elevation of the bottom and top of a lighthouse are found to be $26^\circ 45'$ and $31^\circ 20'$ respectively. The lighthouse is known to be 28.5 feet high. Find the height of the cliff on which it stands. Ans. 137 feet

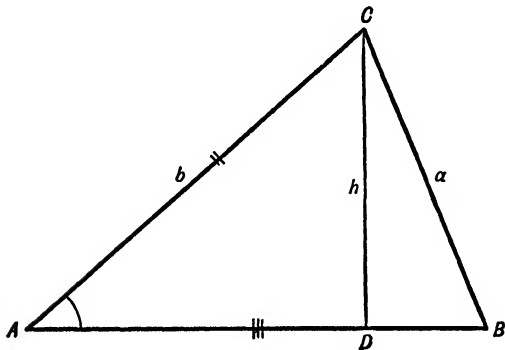


FIG. 43

65. The Law of Cosines.

If none of the ratios in the law of sines can be evaluated, other methods of attack are needed. Let us suppose that A, b, c are the given parts (Figure 43). We wish to express the third side in terms of the given parts. The method is to solve the right triangle ACD for h and AD . After BD is calculated, the

right triangle BCD can be solved.

$$AD = b \cos A$$

$$BD = c - b \cos A$$

Using the Pythagorean relation, we have

$$h^2 = b^2 - AD^2 = b^2 - b^2 \cos^2 A$$

$$a^2 = h^2 + BD^2$$

$$= b^2 - b^2 \cos^2 A + (c - b \cos A)^2$$

$$= b^2 - b^2 \cos^2 A + c^2 - 2bc \cos A + b^2 \cos^2 A$$

Two terms drop out in the right-hand member, resulting in the *law of cosines*

$$a^2 = b^2 + c^2 - 2bc \cos A \quad 11-3$$

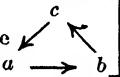
As an aid in remembering the law of cosines, we notice that it resembles very closely the theorem of Pythagoras (equation 3-2); if angle $A = 90^\circ$, then $\cos A = 0$, and the law of cosines reduces to equation 3-2. The

term $2bc \cos A$ may be regarded as a correcting term; if A is less than 90° , the correction must be *subtracted* from the sum of the squares of the two given sides. As the angle becomes smaller, the correcting term becomes larger, because the cosine of an angle increases as the angle decreases. On the other hand, if A is greater than 90° , the cosine is negative, and the correction must be *added* to the sum of the squares of the given sides. In this case, the numerical value of the correcting term increases as the angle increases, because the angle is in the second quadrant.

For problems requiring numerical computation, the law of cosines is clumsy, and its use is avoided unless the given sides are small whole numbers. It is, however, amenable to the operations of algebra, and is frequently employed in theoretical discussions or investigations.

Exercises

- Express in words the meaning of equation 11-3.
- Write the formula for the law of cosines when side b is unknown.

[Suggestion: use the cyclic device .

- Like Exercise 2, when side c is unknown.

66. The Solution of Oblique Triangles, Cases III and IV. When the given parts of a triangle are such that only one factor is known in each of the three ratios of the law of sines, a fourth part must be found by using some other relation before the use of the law of sines is practicable.

Case III. *Given two sides and the included angle.* Let

the two sides be 11 and 14, with the included angle 105° (Figure 44). Then

$$\begin{aligned} a^2 &= 11^2 + 14^2 - 2(11)(14) \cos 105^\circ \\ &= 121 + 196 + 79.8 \end{aligned}$$

$$\therefore a = 20 \text{ (to two significant figures)}$$

The other angles are easily found by the law of sines.

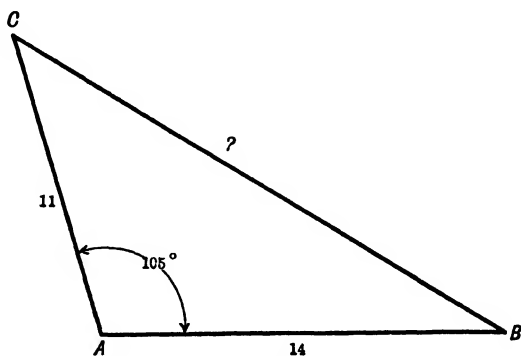


FIG. 44

Again, let the two sides be 4265 and 3228, with the included angle $57^{\circ}39'$ (Figure 45). A straightforward solution by means of right triangles is the following (see pages 25–26):

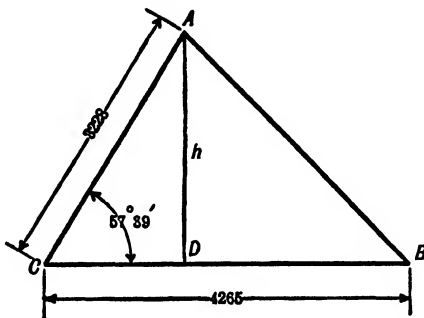


FIG. 45

Solution

$$CD = 3228 \cos 57^{\circ}39'$$

$$= 1727.3$$

$$BD = 4265 - CD$$

$$= 2537.7$$

$$h = 3228 \sin 57^{\circ}39'$$

$$\tan B = \frac{h}{BD}$$

$$B = 47^{\circ}4' \text{ Ans.}$$

$$c = \frac{h}{\sin B}$$

$$= 3725 \text{ Ans.}$$

$$A = 180^{\circ} - (B + C)$$

$$= 75^{\circ}17' \text{ Ans.}$$

Computation

$$\log 3228 = 3.50 \ 893$$

$$\log \cos 57^{\circ}39' = 9.72 \ 843 - 10$$

$$\log CD = 3.23 \ 736$$

$$4265$$

$$(-) 1727.3$$

$$2537.7$$

$$\log 3228 = 3.50 \ 893$$

$$\log \sin 57^{\circ}39' = 9.92 \ 675 - 10$$

$$\log h = 3.43 \ 568$$

$$\log 2537.7 = 3.40 \ 444$$

$$\log \tan B = 0.03 \ 124$$

$$\log h = 3.43 \ 568$$

$$\log \sin 47^{\circ}4' = 9.86 \ 460 - 10$$

$$\log c = 3.57 \ 108$$

$$180^{\circ} = 179^{\circ}60'$$

$$B + C = 104^{\circ}43'$$

$$A = 75^{\circ}17'$$

These results may be checked by the law of sines, using natural functions.

$$\begin{array}{r} A = 75^{\circ}17' \\ B = 47^{\circ}4' \\ C = 57^{\circ}39' \\ \hline A + B + C = 179^{\circ}60' \end{array}$$

$$\frac{a}{\sin A} = \frac{4265}{0.96719} = 4410$$

$$\begin{array}{r} 4265 \quad | 96719 \\ \hline 38687 \quad 4410 \\ 3963 \end{array}$$

$$\frac{b}{\sin B} = \frac{3228}{0.73215} = 4409$$

$$\begin{array}{r} 3228 \quad | 73215 \\ \hline 29286 \quad 4409 \\ 2994 \end{array}$$

$$\frac{c}{\sin C} = \frac{3725}{0.84480} = 4409$$

$$\begin{array}{r} 3725 \quad | 8448 \\ \hline 33792 \quad 4409 \\ 3458 \end{array}$$

Case IV. Given three sides. Problems falling under this case are comparatively rare in technical work. Let the three sides be 43, 53, and 57 (Figure 46). The law of cosines may be expressed in the form

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{53^2 + 57^2 - 43^2}{2(53)(57)} \\ &= \frac{4210}{6040} \\ &= 0.697 \end{aligned}$$

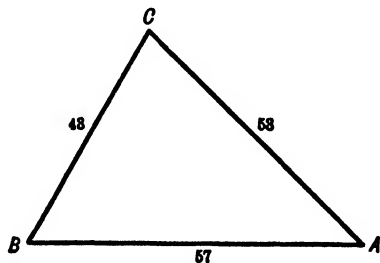


FIG. 46

Hence $A = 45^{\circ}45'$. The other angles are now readily found by the law of sines.

67. Areas. The area of a triangle is known from geometry to be given by one-half the product of the base and the altitude. In Figure 47, the altitude is equal to $a \sin C$; hence the formula

$$\text{Area} = \frac{1}{2}ab \sin C$$

Since the triangle of Figure 47 is general, we may say that *the area of any triangle is equal to one-half the product of any two sides, multiplied by the sine of the included angle.*

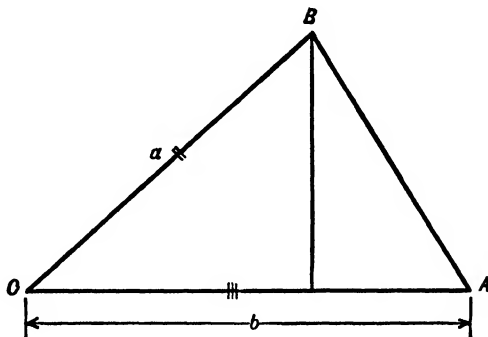


FIG. 47

Exercises

Find the remaining parts, and the area, of each of the following triangles.

- | | | |
|--|--|--|
| 1. $a = 4331$
$b = 5266$
$C = 62^{\circ}44'$ | 2. $A = 96^{\circ}19'$
$b = 43.78$
$c = 55.11$ | $a = 74.06$
$Ans. B = 35^{\circ}59'$
$C = 47^{\circ}42'$ |
| 3. $a = 23\ 579$
$B = 71^{\circ}13.7'$
$c = 31\ 642$ | 4. $a = 513.22$
$b = 743.88$
$C = 44^{\circ}16.7'$ | $A = 43^{\circ}35.2'$
$Ans. B = 92^{\circ}8.1'$
$c = 519.69$ |
| 5. $A = 31^{\circ}45'$
$b = 412$
$c = 377$ | 6. $a = 16.8$
$B = 56^{\circ}30'$
$c = 22.4$ | $A = 47^{\circ}$
$Ans. b = 19.2$
$C = 76^{\circ}30'$ |
| 7. $a = 412$
$b = 513$
$c = 372$ | 8. $a = 61.7$
$b = 55.1$
$c = 63.3$ | $A = 62^{\circ}15'$
$Ans. B = 52^{\circ}15'$
$C = 65^{\circ}30'$ |

68. The Law of Tangents. The formulas and methods discussed thus far in this chapter are those which are most widely used at present in applications of mathematics requiring that one or more oblique triangles be solved.

For special purposes, it is possible to devise formulas and methods that are shorter and more efficient, when applied to the particular kind of problem for which they are designed. Let us write, using the law of sines,

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$

Subtracting 1 from each member,

$$\frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B} \quad 11-6$$

In like manner, by adding 1 to each member of equation 11-5, we have

$$\frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B} \quad 11-7$$

Dividing each member of equation 11-7 by the corresponding member of 11-6, the result is

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

Next, by employing equations 10-18 and 10-19, namely

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

and

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

we obtain

$$\frac{a+b}{a-b} = \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}} \quad 11-8$$

which is the *law of tangents*. It may be used for solving triangles of the side-angle-side type (Case III); and it is preferred by many for logarithmic checking of oblique triangles.

Example. Let the two sides be $a = 43$ and $b = 31$, with the included angle $C = 66^\circ$.

We see that

$$A + B = 180^\circ - C = 114^\circ$$

Hence, from the law of tangents, we have

$$\frac{43+31}{43-31} = \frac{\tan \frac{57^\circ}{2}}{\tan \frac{A-B}{2}}$$

or
$$\tan \frac{A - B}{2} = \frac{12 \tan 57^\circ}{74}$$

$$\frac{A - B}{2} = 14^\circ$$

But

$$\frac{A + B}{2} = 57^\circ$$

Upon adding the last two expressions, we have

$$A = 71^\circ$$

Upon subtracting the same two expressions, we find

$$B = 43^\circ$$

By the law of sines, we now find

$$c = 41.6$$

which should be rounded off to 42.

69. Tangents of the Half-Angles of a Triangle. From the law of cosines, we may write

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$

This reduces, after factoring, to the expression

$$1 - \cos A = \frac{(a + b - c)(a + c - b)}{2bc}$$

Introducing the semi-perimeter

$$s = \frac{1}{2}(a + b + c) \quad 11-9$$

we observe that

$$a + b - c = 2(s - c)$$

$$b + c - a = 2(s - a)$$

$$c + a - b = 2(s - b)$$

Hence we may write

$$1 - \cos A = \frac{2(s - b)(s - c)}{bc} \quad 11-10$$

In like manner, we find that

$$1 + \cos A = \frac{2s(s - a)}{bc} \quad 11-11$$

Now from equations 10-16 and 10-17, we may write

$$\begin{aligned}\tan^2 \frac{A}{2} &= \frac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}} \\ &= \frac{1 - \cos A}{1 + \cos A} \\ &= \frac{(s-b)(s-c)}{s(s-a)}\end{aligned}$$

Introducing a new quantity, defined by the equation

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} \quad 11-12$$

we may write

$$\tan^2 \frac{A}{2} = \frac{r^2}{(s-a)^2}$$

Since the triangle is general, similar formulas result for the other two angles. Thus, finally,

$$\tan \frac{A}{2} = \frac{r}{s-a} \quad \tan \frac{B}{2} = \frac{r}{s-b} \quad \tan \frac{C}{2} = \frac{r}{s-c} \quad 11-13$$

It may be shown that the quantity r represents the radius of the inscribed circle.

The three formulas (11-13) just obtained may be employed in the solution of triangles of the side-side-side type (Case IV).

Example. Let the three sides be 212, 231, and 197.

We first compute

$$s = \frac{1}{2}(212 + 231 + 197) = 320$$

and

$$s-a = 108 \quad s-b = 89 \quad s-c = 123$$

Next, we calculate the value of

$$r = \sqrt{\frac{(108)(89)(123)}{320}} = 60.8$$

Finally, we have

$$\begin{aligned}\tan \frac{A}{2} &= \frac{60.8}{108} & \tan \frac{B}{2} &= \frac{60.8}{89} & \tan \frac{C}{2} &= \frac{60.8}{123} \\ \frac{A}{2} &= 29^{\circ}20' & \frac{B}{2} &= 34^{\circ}20' & \frac{C}{2} &= 26^{\circ}20' \\ A &= 58^{\circ}40' & B &= 68^{\circ}40' & C &= 52^{\circ}40'\end{aligned}$$

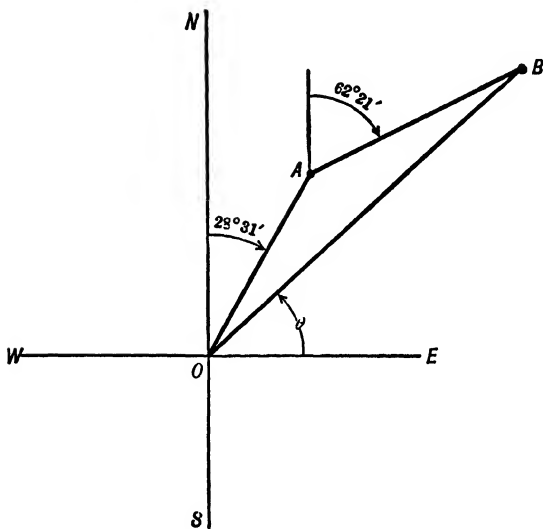


FIG. 48

Exercises

Solve and check, using the method that seems most appropriate.

- Two forces of 2170 lb. and 3360 lb. have a resultant of 1640 lb. Find the angle between the given forces. *Ans.* 156°
- An airplane flying at a constant altitude of 14,700 feet drops a marker which makes an aluminum slick at a point *A* on the ocean beneath. At a certain time the angle of depression of *A* is 41°30', and 50 seconds later it is 24°15'. How far did the airplane move during the 50 seconds?
- In Figure 48, point *A* is distant 728.1 feet from *O*, and bears N 28°31' E. Point *B* is distant 840.9 feet from *A*, and bears N 62°21' E. Find the distance *OB*, and the angle θ made by *OB* with an east-west line. *Ans.* 1501.5, 43°19'

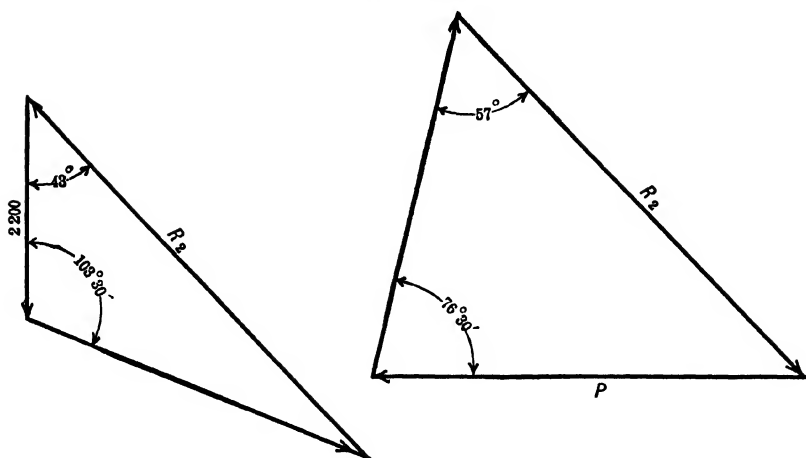


FIG. 49

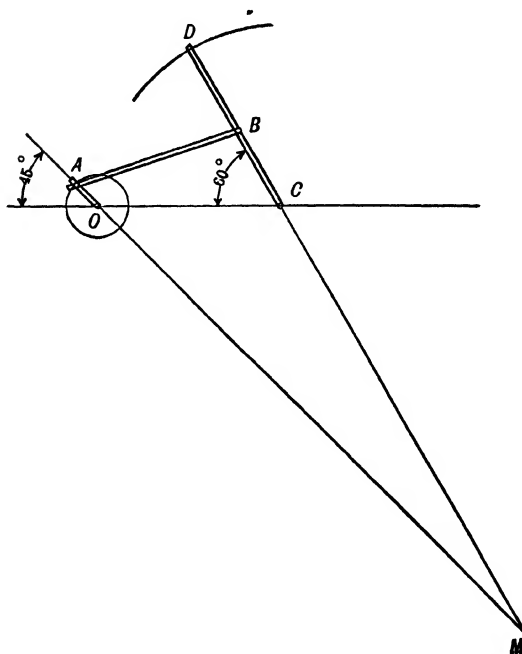


FIG. 50

4. Solve the force triangles shown in Figure 49 for the unknown force P . The reaction R_2 is the same (except in sense) in both triangles.
5. Two rotating arms OA and CD are linked by a coupling rod AB (Figure 50). The lengths OA , AB , and BC are 1, 6, and 3 feet respectively. It is required to find the distances from A and B to M , the instantaneous center of motion.
6. Two sides of a triangle are 7 and 8. The included angle is 120° . Find the remaining parts.
7. If angle $A = 30^\circ$, and side $b = 100$, how many solutions are there if $a = 25$? If $a = 50$? If $a = 75$? If $a = 125$?
8. From a point on the shore, two buoys bear $N 13^\circ 25' E$ and $N 37^\circ 10' E$ respectively. If it is 903 feet to the farther buoy, and the buoys are 495 feet apart, what is the distance to the nearer buoy?
9. N holes are to be drilled into a steel plate, with their centers equally spaced on the circumference of a circle D inches in diameter. Find the straight-line distance between the centers of two consecutive holes.

Ans. 22.6, 20.6

Ans. $D \sin \frac{180^\circ}{N}$

CHAPTER 12

THE QUADRATIC FUNCTION

70. Quadratic Equations. By a quadratic equation is meant any equation of the form

$$ax^2 + bx + c = 0 \qquad 12-1$$

This means that a quadratic equation must contain at least one term involving the *square* of the unknown; it may contain terms involving the *first power* of the unknown; and it may contain terms *not involving* the unknown. These are the only permissible kinds of terms.

A literal solution may be obtained by the method called *completing the square*, employing a device that will be useful elsewhere (Chapter 18). The first step is to put those terms not involving the unknown in the right-hand member, thus:

$$ax^2 + bx = -c$$

Next, let both members be divided by the coefficient of the second degree term:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now comes the key step, completing the square on x . We add to the left-hand member *the square of one-half the coefficient of x* . This is legitimate, if the same quantity is added to the right-hand member as well:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

The left-hand member is now a perfect square.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

The point of these operations now appears. By taking the square root of both members, the problem of solving *one* equation of the *second* degree

is reduced to the solution of *two* equations of the *first* degree:

$$x + \frac{b}{2a} = \frac{+\sqrt{b^2 - 4ac}}{2a}$$

$$x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}$$

From these we obtain the two roots r_1 and r_2 :

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

12-2

That both r_1 and r_2 are roots of equation 12-1 may be verified by direct substitution (see the exercises below).

Example 1. Let us solve the equation

$$7.17x^2 - 0.226x - 0.0311 = 0$$

The method of completing the square works well when the coefficients are decimal quantities, as is frequently the case in applications. Dividing both members by 7.17, and transposing the constant term, we have

$$x - 0.0315x = 0.004 \ 34$$

Adding to both members the square of one-half the coefficient of x ,

$$x^2 - 0.0315x + 0.0158^2 = 0.004 \ 34 + 0.000 \ 25$$

$$(x - 0.0158)^2 = 0.004 \ 59$$

$$x - 0.0158 = \pm 0.0678$$

Hence the roots are

$$r_1 = 0.0836$$

$$r_2 = -0.0520$$

Example 2. Consider the equation

$$x^2 + x - xy - 2y^2 - 2y = 0$$

It is required to solve for x , treating y as a known quantity.

In literal problems, the method of completing the square sometimes proves rather laborious. It is preferable to employ the formulas 12-2. Since

$$a = 1$$

$$b = 1 - y$$

$$c = -2y^2 - 2y$$

we have

$$b^2 - 4ac = 1 + 6y + 9y^2 = (1 + 3y)^2$$

Hence

$$r_1 = \frac{-(1 - y) + (1 + 3y)}{2} = 2y$$

$$r_2 = \frac{-(1 - y) - (1 + 3y)}{2} = -y - 1$$

Exercises

1. Solve the equation

$$5x^2 - 8x - 6 = 0$$

by completing the square, and also by using the quadratic formula.

2. Solve the equation

$$3R^2 - 6R + 2 = 0$$

3. Solve the equation

$$3y^2 + 2xy - x^2 - x - y = 0$$

for x in terms of y .

$$\text{Ans. } 3y - 1, -y$$

4. Solve for x :

$$4.66x^2 - 0.379x - 0.0261 = 0$$

5. Solve for x :

$$x^2 \cos 30^\circ + x \sin 45^\circ = 0.929$$

$$\text{Ans. } 0.705, -1.521$$

6. Determine a, b, c in the equation

$$3x^2 \cos \phi - kx + 2k + x = 0$$

7. Determine a, b, c in the equation

$$2x^2 - x + 3mx + 3m - m^2x^2 = 0$$

8. Solve for x :

$$x(ab + x) = ab(a + 1) + x + ax$$

$$\text{Ans. } a + 1, -ab$$

9. Solve for x :

$$2x^2 + bx + b = 2x + b^2$$

$$\text{Ans. } \frac{b}{2}, 1 - b$$

10. By direct substitution, show that the quantity r_1 satisfies equation 12-1.

11. Show that the quantity r_2 satisfies equation 12-1.

71. Solution by Factoring. If a pair of rational factors of the left-hand member can be found, it is convenient to solve equation 12-1 by the

method of the following example. Suppose

$$3x^2 + 2x - 5 = 0 \quad 12-3$$

In factored form, this becomes

$$(3x + 5)(x - 1) = 0 \quad 12-4$$

Evidently any value of x that reduces the factor $(3x + 5)$ to zero will reduce the entire left-hand member of equation 12-4, and therefore of equation 12-3, to zero. Hence the root of

$$3x + 5 = 0$$

is one root of equation 12-3. Similarly, the root of

$$x - 1 = 0$$

is another root of equation 12-3. The roots are $-\frac{5}{3}$ and $+1$.

The equation

$$(x - r_1)(x - r_2) = 0 \quad 12-5$$

evidently possesses the roots r_1 and r_2 . If these roots are expressed by the formulas 12-2, equation 12-5 is equivalent to equation 12-1.

Exercises

1. Show by expanding equation 12-5 that the product of the roots is equal to $+c/a$.
2. Show that the sum of the roots is equal to $-b/a$.
3. Solve by factoring

$$6x^2 + 11x - 10 = 0 \quad \text{Ans. } \frac{2}{3}, -\frac{5}{2}$$

4. Criticize the following "solution":

$$2x^2 + 4x = 9$$

$$2(x)(x + 2) = 9$$

$$\therefore x = 3 \text{ and } x + 2 = -3$$

5. Solve by factoring

$$2x^2 - 4x + kx^2 = 0 \quad \text{Ans. } 0, \frac{4}{2+k}$$

6. Solve by factoring

$$mx^2 + 17mx - 3x^2 = 0$$

7. What is the quadratic equation whose roots are $\frac{3}{2}$ and -5 ?

$$\text{Ans. } 2x^2 + 7x - 15 = 0$$

8. What is the equation whose roots are $2 \pm \sqrt{3}$?

72. The Nature of the Roots. Certain useful generalizations can be made by an examination of the roots 12-2:

- (a) Every equation of the second degree has just two roots.
 (b) The two roots are *equal* if the quantity $b^2 - 4ac$ is zero.
 (c) The roots are both *rational* if $b^2 - 4ac$ is a perfect square. This is the factorable case.

(d) The roots are *real, unequal and irrational* if $b^2 - 4ac$ is positive, but not a perfect square.

(e) The roots are *complex* (imaginary) if $b^2 - 4ac$ is negative. Complex numbers will be taken up in the next chapter. It should not be assumed that this case is not of practical importance. On the contrary, there are many applications in electricity and mechanics where the existence of complex roots has a definite and important physical meaning.

The quantity $b^2 - 4ac$ is appropriately called the *discriminant* of the equation 12-1.

(f) The sum of the roots is equal to $-b/a$.

(g) The product of the roots is equal to $+c/a$.

73. Equations in the Quadratic Form. Many equations may be reduced to the form 12-1 by a suitable substitution. Thus

$$3 \cos^2 A + 2 \cos A - 5 = 0 \quad 12-6$$

may be reduced to a quadratic equation by the substitution $x = \cos A$.

We may say that equation 12-6 is a quadratic equation *in the cosine of* A . But since equation 12-6 is *not* a quadratic equation in A , the theorems of the previous section do not apply; there are in fact an infinite number of roots (see Chapter 14).

Exercises

Solve the following equations.

1. $8(y - 3a)^2 + 37(y - 3a) - 15 = 0$

Ans. $3a - 5, \frac{24a + 3}{8}$

2. $3\left(\frac{2}{y-1}\right)^2 - 5\left(\frac{2}{y-1}\right) + 2 = 0$

Ans. $\pm\sqrt{3}, \pm\frac{1}{\sqrt{2}}$

3. $2t^4 - 7t^2 + 3 = 0$

4. $10z - 27z^{1/2} + 5 = 0$

5. $e^{2x} - 5e^x + 6 = 0$

Ans. $\ln 2, \ln 3$

6. $e^{2x} = 5e^x + 24$

7. $\log x + \log(x + 6) = 2 \log(3x - 2)$

Ans. 2

8. $\log(4x + 2) - 2 \log(2x + 1) + \log(6x - 7) = 0$

9. $\sqrt{1 + y} = 4 - \sqrt{17 + 4y}$

Ans. $-\frac{1}{2}$

10. $\sqrt{2y + 1} = 1 - \sqrt{y}$

11. The radius of a cylindrical tank is 3 feet and the altitude is also 3 feet. By what amount may the radius be decreased, and the altitude increased, without changing the volume? *Ans.* $\frac{3 + \sqrt{45}}{2}$

12. Solve Exercise 11, if the radius and altitude are both h feet in length.

74. The Quadratic Function. The theory of quadratic equations is a part of the theory of the quadratic function

$$f(x) = ax^2 + bx + c \qquad \mathbf{12-7}$$

It will be shown in Chapter 18 that the graph of this function is a parabola.

The quadratic function obviously includes the linear function as a special case (when $a = 0$). When the linear function gives only a first approximation to a physical law, a closer approximation may frequently be obtained by employing the function 12-7. The problem of calculating the coefficients a , b , and c , when the value of the function is known for three distinct values of the argument, was discussed in Chapter 7.

Exercises

1. Graph the function

$$f(x) = 3x^2 - 2x - 5$$

by constructing a table of values of $f(x)$ for values of x lying between -3 and $+3$. Hence solve graphically the equations

(a) $3x^2 - 2x - 5 = 0$

(b) $3x^2 - 2x - 2 = 0$

2. When the discriminant $b^2 - 4ac$ is negative, is it possible for the graph of the function 12-7 to cross the x -axis?
3. Solve the equation

$$\sqrt{V_1^2 - V^2} + \frac{\sqrt{w}}{2} = 2\sqrt{w}$$

for V in terms of the other letters.

Ans. $\pm \frac{1}{2} \sqrt{4V_1^2 - 9w}$

4. The maximum stress P of a machine member, subjected to a suddenly applied load W , is found by solving the equation

$$P^2 = \frac{2W^2h}{L} + 2WP$$

Find P in terms of the other letters.

Ans. $W \left(1 + \sqrt{1 + \frac{2h}{L}} \right)$

5. A long strip of zinc, 25 inches wide, is to be made into a gutter by turning strips up vertically along the two sides. How many inches should be turned up to give the rectangular cross-section an area of 70 square inches?
- Ans.* 4.23 or 8.27
6. The sum of the roots of a quadratic equation is $-\frac{1}{2}$. The product is -3 . What is the equation?

7. (a) Solve for x in terms of y :

$$x^2 - xy - 2y^2 + x + 4y - 2 = 0$$

- (b) Use your answers to factor the left-hand member.

Ans. $(x - 2y + 2)(x + y - 1)$

8. A quantity of coal is piled on the ground, forming a cone 12 feet high, whose slant height is inclined at 34° with the horizontal. The coal is to be transferred to a trapezoidal bin 20 feet long, with the cross-section shown in Figure 51. To what depth will the bin be filled?

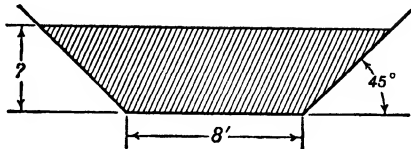


FIG. 51

9. In finding the curve which best fits a table of experimental data, it is necessary to solve an equation of the form

$$m^2 - \left(A - \frac{1}{B}\right)m - 1 = 0$$

- (a) Find m if $A = B$.

Ans. $A, -\frac{1}{A}$

- (b) Find m if $A = 1.60$ and $B = 1.50$.

Ans. $1.571, -0.637$

- (c) Check the answers to part (b) by finding the sum and product of the roots.

Miscellaneous Problems

1. If $f(x) = \frac{1 - x^2}{1 + x^2}$, show that $f(-x) = f(x)$, and that $f\left(\frac{1}{x}\right) = -f(x)$.

2. If

$$\frac{a}{b} = \frac{c}{d}$$

show that

$$\frac{a + kb}{b} = \frac{c + kd}{d}$$

3. If $y - b$ is directly proportional to x , where b represents a constant known in advance, what equation connects y and x ? What is the nature of the graph represented by this equation?

4. Solve the equation

$$r = c(k + r \cos \theta)$$

for r in terms of the other letters.

5. Write as a single power of a :

$$\sqrt{\frac{1}{a^3}}$$

6. Write with a single exponent:

$$10^{8.7-10}$$

$$10^{8.71-10}$$

$$10^{8.712-10}$$

$$10^{8.7127-10}$$

7. The force necessary to raise a heavy load by means of a jackscrew is found by means of the simultaneous equations

$$\begin{cases} W + \mu \sin A \sum N = \cos A \sum N \\ Pa = r \sin A \sum N + \mu r \cos A \sum N \end{cases}$$

Eliminate the quantity $\sum N$, and solve for P . Set $\mu = \tan \phi$, and simplify.

8. The number of foot-pounds of work done during one cycle in an air compressor is calculated from the formula

$$W = \frac{26\ 500\ n}{1 - n} \sqrt{\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1}$$

Calculate W for $p_1 = 100$, $p_2 = 15$, and $n = 1.34$.

9. Solve for x :

$$x^2 + a^2 + x - a = 2ax$$

10. A polygon of M sides always has $\frac{M(M-3)}{2}$ diagonals. How many sides has a polygon of 135 diagonals?

11. Remove the radical and solve for m in terms of the other letters:

$$y = \sqrt{s^2 - m^2} - m$$

(This equation occurs in the theory of the catenary, the curve assumed by a suspended cable or wire.)

12. Solve for x :

$$\frac{a^2x^{-2} - 1}{2x^{-1}} = \frac{3}{2}$$

13. For what values of K will the roots of the following equation be equal?

$$x^2 - 7x + Kx + 49 = 0$$

14. Solve the equation

$$T = 2\pi rh + 2\pi r^2$$

for r in terms of the other quantities.

15. Solve for x :

$$\frac{x^{1/2} - s}{2 - 2x^{1/2}} = \frac{3x^{1/2}}{7}$$

16. The ionization constant of a weak acid is given by the formula

$$K = \frac{x^2}{(1-x)v}$$

What is the value of x in terms of K and v ?

17. The density of water at $\theta^\circ \text{C}$. is given by the empirical formula

$$d = 1 - \frac{96(\theta - 4)^2}{10^7}$$

Find θ for $d = 0.999\ 971$.

18. In the design of a built-up column there is encountered the formula

$$I_1 = I_2 + a(b+c)^2$$

Solve for b in terms of the other letters.

19. If $\tan A = \frac{1}{x}$, find $\sin(90^\circ - A)$.

20. The moment that can be transmitted by a single-cone clutch is

$$M = \frac{\pi \mu c}{\sin A} (r_2^2 - r_1^2)$$

where r_1 and r_2 are the inner and outer radii of a conical disc, and c is a constant whose value depends upon the normal wear of the friction surfaces. The axial force P is

$$P = 2\pi c(r_2 - r_1)$$

Set

$$D = r_1 + r_2$$

and express M in terms of P , μ , $\sin A$, and D .

21. Transform the expression $\frac{\csc A}{\cot A}$ into an equivalent expression containing no other function than the sine.

22. A shaft of diameter 3.375 inches is to have a square keyway 0.375 inches high cut along it (Figure 52). What is the additional distance which must be added to 0.375 inches to give the total depth from the point where the cutting tool first begins to cut?

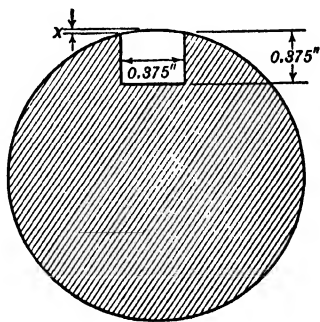


FIG. 52

23. Two forces, one of 455 lb. and the other of 387 lb., make an angle of $53^\circ 20'$. Find the magnitude and direction of their resultant.

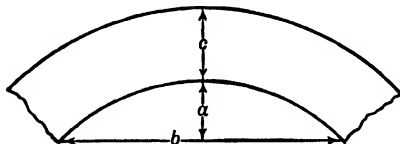


FIG. 53

24. A section of a damaged rim was measured in order to construct a new one. The dimensions were found to be (see Figure 53): $a = 3.00$ inches (taken at the middle); $b = 12.00$ inches; and $c = 1.25$ inches. Find the outside diameter. What percentage of the rim was missing?

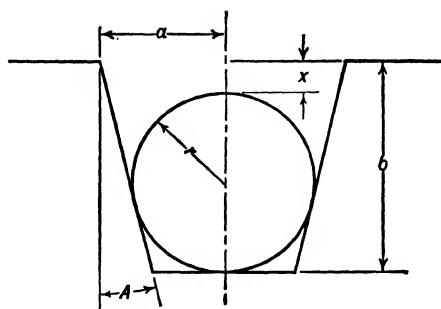


FIG. 54

25. A tapered keyway is to be ground into a machine housing. To check the sizes, a measuring wire is inserted, of such size that it is tangent to the three sides as shown in Figure 54, and the distance x is measured. Find r and x , if $a = 0.84$ inches, $b = 1.50$ inches, and $A = 7^\circ$. Show that

$$r = \frac{a \cos A - b \sin A}{1 - \sin A}$$

26. One end of a connecting rod AB , 5 feet long, is fastened to a crank BO , 1 foot long, while the other end is fastened to a crosshead A which is constrained to move along AO . (Figure 55.) How far from the extreme position P of the crosshead will A be when OB is perpendicular to AB ? When angle BOA is 60° ?

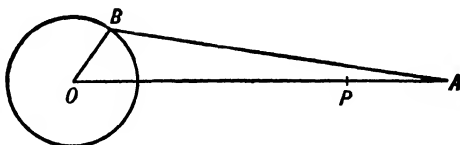


FIG. 55

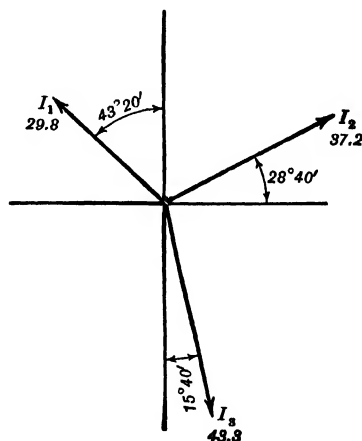
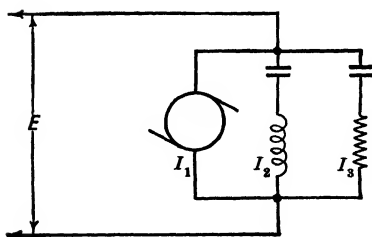


FIG. 56

27. A preliminary calculation shows that the currents through a divided (parallel) circuit may be represented by the vectors shown in Figure 56. Find the resultant current (the vector sum of I_1 , I_2 , I_3).

28. A survey is run around a mountain, through which a tunnel is to be constructed. A plan of the work is shown in Figure 57. What is the distance and bearing of point D from A ?

Point	Distance	Bearing
A	423.62	N 27-15-10 E
B	376.21	N 81-19-40 E
C	192.07	S 47-26-00 E
D	321.74	S 15-45-50 E

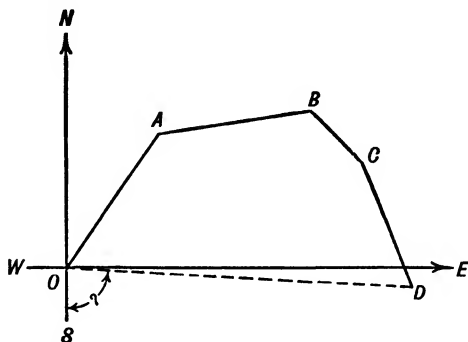


FIG. 57

29. Show that the area of a quadrilateral is equal to one-half the product of its diagonals by the sine of the included angle.
30. Simplify the formula

$$\tan A_1 = \frac{\sin (180^\circ - \theta)}{\frac{T_2}{T_1} - \cos (180^\circ - \theta)}$$

(This equation gives the center angle of an obtuse-angle bevel gear.)

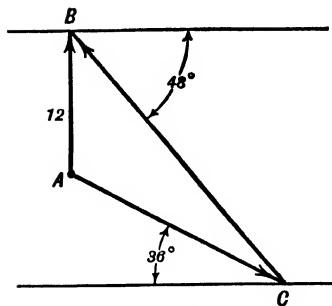


FIG. 58

31. In Figure 58, AB is a vertical force. It is resolved into components AC and BC which make angles of 36° and 48° , respectively, with the horizontal. Find the magnitudes of these components.

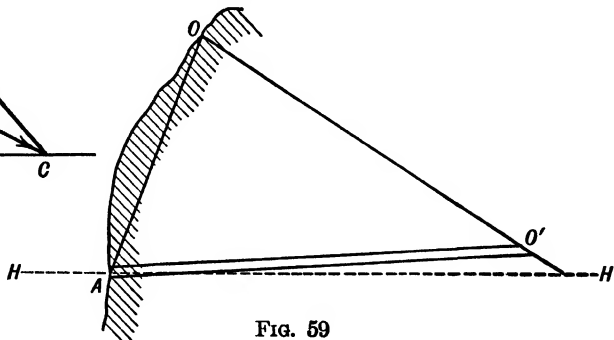


FIG. 59

32. In Figure 59, line OO' traces the plane of a vein of ore, outcropping at O on a hillside. It makes an angle of 33° with the horizontal $H-H$. A tunnel AO' is to be driven into the hill to tap the vein. The tunnel slopes upward at an angle

of 8° to the horizontal. If AO is 740 feet long, and makes an angle of 41° with the horizontal, how long will the tunnel be?

33. In Figure 60, the length of AB may be taken to be 1 unit.
 (a) Express BD as a function of three of the given angles.
 (b) Find a formula for $\log BC$ in terms of three of the given angles.
 (c) Express CD in terms of BD , BC , and z .

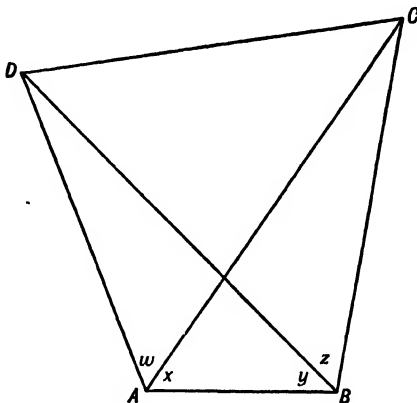


FIG. 60

34. A certain magnetic field H is the resultant of two fields H_1 and H_2 (Figure 61).

H_1 = magnetic field due to current I in first conductor.

H_2 = magnetic field due to current I in second conductor.

d_1 = distance from the first conductor.

d_2 = distance from the second conductor.

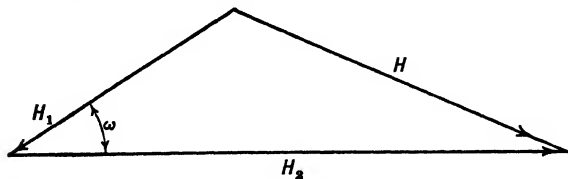


FIG. 61

It is known that

$$H_1 = \frac{0.2I}{d_1}$$

$$H_2 = \frac{0.2I}{d_2}$$

Find a formula expressing H in terms of d_1 , d_2 , I , and ω .

35. Find $\tan 285^\circ$ without reference to tables, and express your answer in simplest radical form.

36. Eliminate P from the simultaneous equations

$$\begin{cases} P \cos \theta - W \sin A = 0 \\ R + P \sin \theta - W \cos A = 0 \end{cases}$$

and solve for R , simplifying your answer by means of the addition theorems of trigonometry.

37. Simplify the expression

$$d_1^2 = \frac{L^2}{16} \sin^2 2B + \left[\frac{L}{4} (1 + \cos 2B) - \frac{2L}{4} \right]^2$$

38. Simplify

$$\tan (45^\circ + A) - \tan (45^\circ - A)$$

39. Let sides b and c , and angle A , of an oblique triangle be given. From the law of sines

$$\frac{b}{\sin (180^\circ - A - C)} = \frac{c}{\sin C}$$

Expand and simplify, and find an equation for $\cot C$ in terms of the given quantities.

40. If $x = R \tan \frac{A}{2}$, and $\tan A = \frac{d}{h}$, show that

$$x = \frac{R}{d} (\sqrt{h^2 + d^2} - h)$$

41. From the simultaneous equations

$$\begin{cases} V = W \sin B \\ P \cos B = W \sin B \end{cases}$$

derive the formula

$$\frac{1}{V^2} = \frac{1}{P^2} + \frac{1}{W^2}$$

CHAPTER 13

COMPLEX NUMBERS

75. Introduction. It is convenient to know that a quadratic equation always has just two roots. However, a simple example like

$$x^2 + 1 = 0$$

shows that the theorem is not true in all cases, *unless we are prepared to consider the existence of numbers whose squares are negative.*

Imaginary numbers were introduced into mathematics because it was found that, by their use, the theorems of algebra could be expressed in a simpler way than would otherwise be possible. Later, when familiarity had conditioned men's minds to acceptance of the queer new numbers, they were found to be a useful tool in physical science.

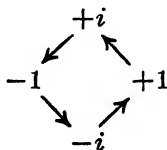
76. The Number i . The set of real numbers described in Chapter 8 is characterized by the fact that *the square of every number in the set is positive.* Let us define a new number i by the equation

$$i^2 = -1 \qquad \qquad \qquad 13-1$$

It is reasonable to require that this number, like its predecessors in the real number system, shall be subject to the laws of exponents. Accordingly, we discover that

$$\begin{aligned} i &= i \\ i^2 &= -1 \\ i^3 &= i^2(i) = -i \\ i^4 &= i^2(i^2) = +1 \\ i^5 &= i^4(i) = i \\ i^6 &= i^5(i) = i^2 \end{aligned} \qquad \qquad \qquad 13-2$$

Higher powers merely continue the cycle



Thus all of the numbers obtained by raising i to an integral power may be expressed in terms of ordinary real numbers and the new number i .

Moreover, any number whose square is negative may be expressed in the form

$$bi \quad 13-3$$

where b is some real number. Since

$$b^2 i^2 = b^2(-1) = -b^2$$

we may write

$$\sqrt{-b^2} = bi$$

or

$$\sqrt{-k} = \sqrt{k}i$$

provided that k is positive. The form 13-3 is called a pure imaginary; that is, a number whose square is negative is a pure imaginary.

77. Complex Numbers. Let us next consider the equation

$$x^2 + x + 1 = 0 \quad 13-4$$

The roots of this equation are usually designated as ω and ω^2 . Upon solving, we find that

$$\begin{aligned} \omega &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ \omega^2 &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{aligned} \quad 13-5$$

The quantities ω and ω^2 are called complex numbers. A complex number is one that can be expressed in the form

$$a + bi \quad 13-6$$

where a and b are in the set of real numbers. It is possible for a or b to take on the value 0. Hence, all real numbers are included in the set of complex numbers as a special case; the pure imaginaries are included as another special case.

In algebraic operations with complex numbers, the quantity i is treated like any other letter, except that i^2 may always be replaced by -1 .

In *adding* (or *subtracting*) two complex numbers, the real parts and the imaginary parts are added (or subtracted) separately. For example,

$$(3 + 2i) + (4 - 5i) = (3 + 4) + (2 - 5)i = 7 - 3i$$

$$(3 + 2i) - (4 - 5i) = (3 - 4) + (2 + 5)i = -1 + 7i$$

Complex numbers are *multiplied* in the ordinary way, except that i^2 is replaced by -1 . For example,

$$\begin{aligned}(3 + 2i)(4 - 5i) &= (3)(4) - (3)(5i) + (2i)(4) - (2i)(5i) \\ &= 12 - 15i + 8i - 10i^2 \\ &= 22 - 7i\end{aligned}$$

The *division* of complex numbers is carried out indirectly, by operating upon the denominator to transform it to a real number. For example,

$$\begin{aligned}\frac{3 + 2i}{4 - 5i} &= \left(\frac{3 + 2i}{4 - 5i}\right) \left(\frac{4 + 5i}{4 + 5i}\right) \\ &= \frac{2 + 23i}{41}\end{aligned}$$

The quantity $(4 + 5i)$ is the *conjugate* of $(4 - 5i)$. By definition, the conjugate of a complex number is another complex number, having the same real part as the given number, and an imaginary part which differs only in sign. We may say that *division of complex numbers is accomplished by multiplying numerator and denominator of the fraction by the conjugate of the denominator*.

It is easily shown that *if two complex quantities are equal, the real parts must be equal, and the imaginary parts must be equal*. For, let the complex expressions be reduced to the forms $(a + bi)$, and $(c + di)$. We wish to prove that $a = c$ and $b = d$.

By hypothesis

$$a + bi = c + di$$

Hence we may write

$$a - c = (d - b)i$$

If d and b are unequal, both members may be divided by $(d - b)$, which gives

$$i = \frac{a - c}{d - b}$$

which is a real number. The assumption that d and b are unequal leads to a contradiction. Hence we conclude that $d = b$. It follows immediately that $a = c$.

78. The Number System of Algebra. The foregoing operations upon complex numbers have in every case resulted in another complex number. It is natural to inquire whether the operations of algebra, when applied to

complex numbers, may not upon occasion generate yet other new numbers. In particular, we have seen that the operation of extracting a root, when applied to rational numbers, led to irrational numbers and imaginary numbers. Might it not prove necessary to introduce other new numbers, in order to interpret the result of extracting a root of a complex number? The answer, which is given without proof, is negative. *Any quantity that can be expressed in terms of a finite number of algebraic operations upon complex numbers may be written in the form $a + bi$.*

Thus, in arriving at the complex number system, we have carried to a logical conclusion the process of generalization. No other numbers can be found, unless the rules of algebra be changed. Let us glance briefly at the structure of the number system.

In elementary arithmetic, positive numbers are dealt with; it is not possible to subtract a larger number from a smaller one. The numbers of arithmetic range upward from zero. But in algebra, in order to generalize the operation of subtraction, negative numbers are admitted to the number system, ranging downward from zero. At first, even mathematicians regarded negative numbers as fictitious. How could there be numbers below zero? But negative numbers were found useful in representing the idea of profit and loss, of distance east and distance west, and so on. In the same way, imaginary numbers were invented in order to generalize the operation of extracting roots. Their existence is neither more nor less actual than the existence of the number 13.

The complex number system consists of all numbers of the form $a + bi$. Setting $b = 0$, we see that the *real numbers* form a subset, contained within the set of complex numbers. In like manner, the *rational numbers* are contained within the set of real numbers. Finally, at the center of this collection of Chinese boxes, we find the set of whole numbers.

Exercises

1. Reduce to the form bi :

(a) $\sqrt{-75}$

(b) $\sqrt{-\frac{1}{4}}$

(c) $\sqrt{b-a}$

2. Show that $\omega^3 = (\omega)(\omega)$ (see equations 13-5).

3. Prove by direct calculation that

$$1 + \omega + \omega^2 = 0$$

4. Reduce to the form $a + bi$:

(a) $\frac{3 - \sqrt{-6}}{3 + \sqrt{-6}}$

(b) $\frac{2+i}{1+2i} - \frac{1}{i}$

(c) $\left(\frac{i}{1+i}\right)^3$

(d) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$

5. Reduce to the form $a + bi$:

(a) $(\sqrt{-2})(\sqrt{-2})$

Ans. -2

(b) $\frac{\sqrt{6}}{\sqrt{-2}}$

Ans. $-\sqrt{3}i$

(c) $\frac{4 + \sqrt{-5}}{3 - 2\sqrt{-5}}$

Ans. $\frac{2}{29} + \frac{11\sqrt{5}}{29}i$

(d) $8i(3\sqrt{-2})(\sqrt{-3})(2i)^3$

Ans. $-192\sqrt{6}$

6. Reduce to the form $a + bi$:

(a) $\frac{1}{\frac{1}{2} + \frac{\sqrt{-3}}{2}}$

(b) $\left(\frac{-1 - \sqrt{-3}}{2}\right)^2$

(c) $\left(\frac{2}{1 - \sqrt{-3}}\right)^2$

7. Find x and y from the equation

$$(1 + x)(1 - i) + (1 + i)(y - \sqrt{2}) = k(1 - i)$$

Ans. $k - 1, \sqrt{2}$

8. If $F(x) = \frac{1}{x^2}$, show that $F(a + bi)$ and $F(a - bi)$ are complex conjugates.

Ans. $F(a + bi) = \frac{a^2 - b^2 - 2abi}{(a^2 + b^2)^2}$

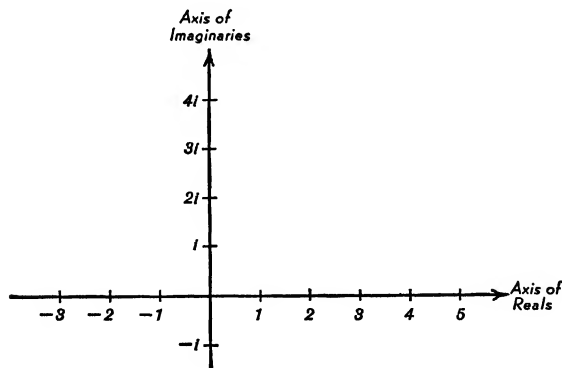


FIG. 62

79. Vector Representation of Complex Numbers. Let us draw two lines at right angles to one another (Figure 62). Points along the horizontal axis are made to correspond to the real numbers, as in Chapter 8. Points on the vertical axis correspond to real multiples of i , that is, to pure imaginary numbers.

Any complex number can be represented by a vector, as shown in Figure 63. The horizontal component of the complex vector is the real part.

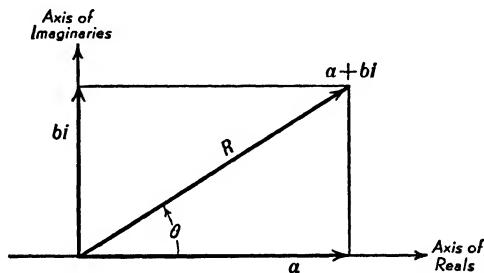


FIG. 63

The vertical component is the imaginary part. The length of the vector is called the *absolute value* of the complex number.

80. Addition and Subtraction of Complex Vectors. The operations of addition and subtraction are performed graphically by means of the

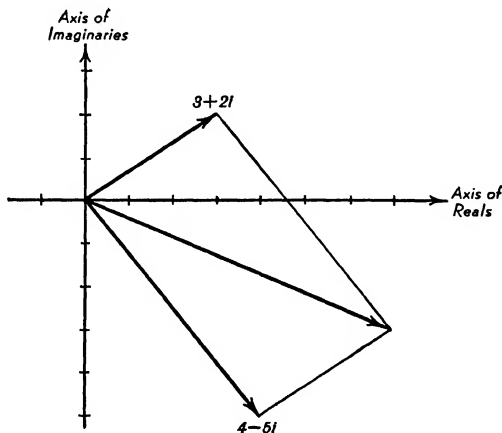


FIG. 64

parallelogram law (Chapter 1). Figure 64 represents the graphical addition of $(3 + 2i)$ and $(4 - 5i)$. Figure 65 represents the graphical subtraction of $(4 - 5i)$ from $(3 + 2i)$.

It should be remarked that subtraction is carried out graphically by reversing the vector to be subtracted (which changes the sign of the vector), and adding.

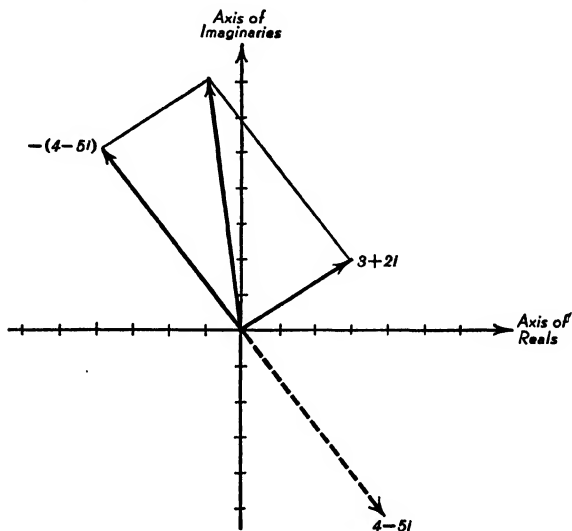


FIG. 65

81. The Polar Form. When the real and imaginary parts are known, the absolute value and the angle are found by means of the equations

$$R = \sqrt{a^2 + b^2} \quad 13-7$$

$$\tan \theta = \frac{b}{a}$$

If the absolute value and angle of the complex vector are known, the real and imaginary parts are found by means of the equations

$$\begin{aligned} a &= R \cos \theta \\ b &= R \sin \theta \end{aligned} \quad 13-8$$

These last equations suggest that a complex vector may be written in the form

$$R(\cos \theta + i \sin \theta) \quad 13-9$$

which is called the polar form.

We shall next show that, in the multiplication of two complex vectors, *the lengths are multiplied, and the angles are added*. Let us take any two complex vectors in the polar form

$$R_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad R_2(\cos \theta_2 + i \sin \theta_2)$$

The product is

$$\begin{aligned} & R_1(\cos \theta_1 + i \sin \theta_1)(R_2)(\cos \theta_2 + i \sin \theta_2) \\ &= R_1 R_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= R_1 R_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] \end{aligned} \quad 13-10$$

The product is thus in the form 13-9, where

$$\begin{aligned} R &= R_1 R_2 \\ \theta &= \theta_1 + \theta_2 \end{aligned}$$

Hence the absolute value of the product of two complex vectors is the product of the absolute values; and the angle of the product of two complex vectors is the sum of the angles of the factors.

Example. We have already found by algebraic methods that

$$(3 + 2i)(4 - 5i) = 22 - 7i$$

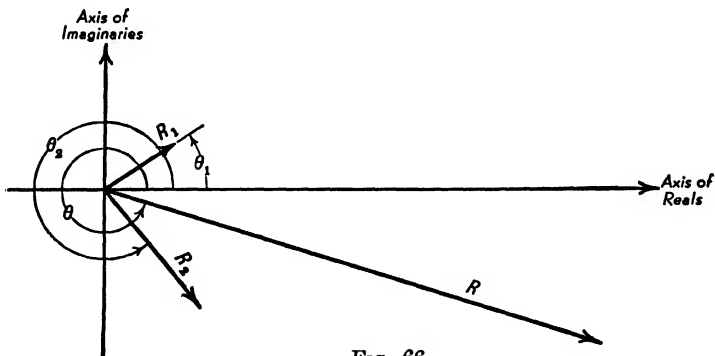


FIG. 66

The two factors and their product are represented in Figure 66. The lengths of the three vectors are

$$\begin{aligned} R_1 &= \sqrt{3^2 + 2^2} = \sqrt{13} \\ R_2 &= \sqrt{4^2 + 5^2} = \sqrt{41} \\ R &= \sqrt{22^2 + 7^2} = \sqrt{533} \\ &= R_1 R_2 \end{aligned}$$

The angles of the three vectors are

$$\theta_1 = 33^\circ 41.4'$$

$$\theta_2 = 308^\circ 39.6'$$

$$\theta = 342^\circ 21.0'$$

$$= \theta_1 + \theta_2$$

For the inverse operation, we have the corresponding theorems:

The absolute value of the quotient of two complex vectors is equal to the quotient of the absolute values of the two vectors.

The angle of the quotient of two complex vectors is equal to the difference of the angles of the two vectors.

82. Vector Operators. The complex vector -1 has the length 1, and the angle 180° (Figure 65). Hence multiplying any complex vector by -1 has the effect of rotating the vector through 180° . This is illustrated in Figure 65, where the subtraction has been carried out by multiplying the vector $(4 - 5i)$ by -1 , and adding.

The complex vector i has the length 1, and the angle 90° . Multiplying any complex vector by i has the effect of rotating the vector through 90° .

The complex vector ω (equation 13-5) has the length 1 and the angle 120° . Multiplying any complex vector by ω has the effect of rotating the vector through 120° .

The quantities -1 , i , ω may be thought of as *operators*, producing the effects described above upon vectors upon which they operate. The representation of complex numbers by vectors reveals a link between algebra and trigonometry, the importance of which it would be difficult to overestimate. This connection is systematically explored, from the standpoint of the mathematician, in the subject called "theory of functions of a complex variable."

Exercises

1. Represent the following vectors graphically. Calculate the length and the angle of each.

(a) $-2 + 7i$	(b) $-3 - 3i$	(c) $3 - 2i$
---------------	---------------	--------------
2. If $a + bi$ and $c + di$ represent (in rectangular form) any two complex vectors, prove that the absolute value of the product is equal to the product of the absolute values.
3. Perform the following operations graphically, and check by algebra.

(a) $(3 - 4i) + (-2 + 5i)$	(b) $(k + 2ki) - (3k - 2ki)$
----------------------------	------------------------------
4. Add graphically $5 + 2i$ and $-2 + 3i$.

5. Subtract $6 + i$ from $-3 + 5i$ graphically.
 6. Add graphically $-3 + 4i$, $2 + 3i$, and $6 - 2i$.
 7. Perform the indicated multiplication graphically, and check by algebra.

(a) $(1 + i)(3 - 4i)$

(b) $(-5 + \sqrt{11}i)(\sqrt{5} + 2i)$

8. Perform the indicated division graphically, and check by algebra.

(a) $\frac{-5 - 12i}{i}$

(b) $\frac{3 - 4i}{\frac{1}{2} - \frac{\sqrt{3}i}{2}}$

(c) $\frac{-4 + 4i}{1 - i}$

9. (a) If A and B are any two complex numbers, show that the angle of the quotient $Q = \frac{A}{B}$ equals the difference in the angles of A and B by considering the product $A = BQ$.

(b) Similarly for the absolute value of $Q = \frac{A}{B}$.

10. Write the following vectors in polar form.

(a) $2 + 2i$

(b) $-1 - \sqrt{3}i$

(c) $-1 + \sqrt{3}i$

11. (a) By factoring $(x^3 - 1)$ into $(x - 1)(x^2 + x + 1)$, show that the three cube roots of unity, that is, the roots of $x^3 - 1 = 0$, are 1 , ω and ω^2 . Draw the vectors representing these three roots on one pair of complex axes.
 (b) The same for $x^4 - 1 = 0$.

83. Powers and Roots. By repeated application of equation 13-10, it may be seen that the product of several complex numbers is a number whose angle is the sum of the angles of the factors, and whose length is the product of the lengths of the factors. Now if the factors be equal, we find that

$$[R(\cos \theta + i \sin \theta)]^n = R^n(\cos n\theta + i \sin n\theta) \quad 13-11$$

That is, *the n -th power of a complex number A is another complex number B . The angle of B is n times the angle of A . The length of the vector B is the n -th power of the length of the vector A .*

The operation of extracting roots is less simple. There are in fact n distinct n -th roots of every complex number (other than zero). For, from equation 13-11, we may write

$$\left[R^{\frac{1}{n}} \left(\cos \frac{\theta + 2\pi k}{n} + i \sin \frac{\theta + 2\pi k}{n} \right) \right]^n = R(\cos \theta + i \sin \theta) \quad 13-12$$

for $k = 0, 1, 2, \dots, n - 1$. For each of the foregoing values of k , the bracketed quantity in the left-hand member of equation 13-12 represents an n -th root of the complex number which is the right-hand member. These roots are equally spaced on a circle of radius $\sqrt[n]{R}$.

A case of special interest is that in which we are calculating a root of a real number, N . If N is positive, there is always a real, positive n -th root, called the principal root. The symbol $\sqrt[n]{N}$ refers to the principal root. If N is negative, and n is odd, the principal root is taken to be that one which is real and negative. If N is negative, and n is even, the principal root is that one having the smallest angle.

For example, $\sqrt[3]{-125} = -5$. This is the principal cube root of -125 ; the others are -5ω and $-5\omega^2$. As another example, $\sqrt{-4} = 2i$. The other square root of -4 is $-2i$. Observe that

$$\begin{aligned}(\sqrt{-a})(\sqrt{-b}) &= (\sqrt{ai})(\sqrt{bi}) \\ &= -\sqrt{ab}\end{aligned}$$

following the convention of selecting the principal roots.

Exercises

1. Find both roots of the equation

$$x^2 = i$$

and draw the vectors representing each.

2. Find all of the roots of the equation

$$x^5 = 32$$

and draw the vectors representing each.

3. Find the value of

$$(\sqrt{3} + i)^6$$

Ans. -64

4. Point out the fallacy in the following argument:

$$(\sqrt{-2})(\sqrt{-3}) = \sqrt{(-2)(-3)} = \sqrt{6}$$

CHAPTER 14

ALGEBRAIC AND TRIGONOMETRIC EQUATIONS

84. The Nature of the Problem. This physical world in which we live has a way of confronting us with formidable mathematical difficulties when we attempt to find an answer to some problem which, to all appearance, is exceedingly simple. For example, a wooden ball is placed in water. The ball has no difficulty in deciding how deep in the water it will float. It is only we who experience difficulty in making the calculation.

The extent to which we rely upon measurements in the field or in the laboratory depends upon our state of ignorance about the nature of the things we measure. When a satisfactory theory is available, only two or three basic measurements need be made, and other necessary quantities are computed from these.

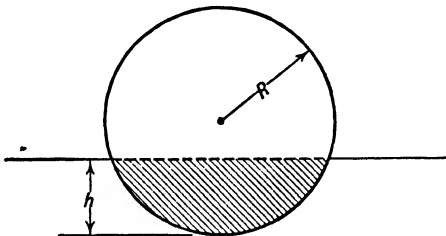


FIG. 67

We resort to direct measurement if no mathematical theory exists, or if the labor of computation exceeds the labor of measuring. The latter situation is unfortunately only too common. It is, of course, aggravated by the use of inefficient computation techniques.

In the problem of the floating wooden ball, we shall find that a well rounded mathematical theory exists. The first step is to arrive at a mathematical formulation of the physical facts. Archimedes' principle informs us that the weight of displaced water is equal to the weight of the object. But the weight of the displaced water equals the volume of the spherical segment of height h (Figure 67), multiplied by the density of water:

$$\text{wt. of displaced water} = \frac{\pi h^2}{3} (3R - h)(\delta_{\text{water}})$$

The weight of the wooden ball equals the volume multiplied by the density:

$$\text{wt. of wooden ball} = \frac{4\pi R^3}{3} (\delta_{\text{ball}})$$

Thus Archimedes' principle leads to the equation

$$\frac{4}{3} \pi R^3 \delta_b = \frac{\pi h^2}{3} (3R - h) \delta_w$$

in which the only unknown quantity is the height h of the spherical segment; that is, the depth of water required to float the ball.

The problem has been formulated; but to find the answer we are seeking, it is necessary to solve an equation of the third degree. Although many methods have been devised for solving cubic equations, there exist no general methods comparable in brevity and simplicity to those available for quadratic equations. Let us begin our investigation of algebraic equations of degree higher than the second by attempting to discover something of the nature of such equations.

85. The Polynomial Function. By algebraic operations such as clearing fractions, squaring to remove radicals, and the like, the solution of most algebraic equations may be reduced to finding the zeros* of a polynomial function of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n \quad 14-1$$

where the coefficients a_0, a_1, \dots, a_n are whole numbers. The polynomial 14-1 evidently includes the linear function 6-1, and the quadratic function 12-7, as special cases. The following treatment will suggest something of the methods and spirit of the vast body of theory, dealing with the properties of the function 14-1, that is in existence.

86. The Remainder Theorem. *If a polynomial function be divided by any linear factor $x - b$, the remainder is numerically equal to the value which the function takes on for $x = b$.* To prove this, we first notice that the result of dividing the polynomial 14-1 by $x - b$ will be a new polynomial, of degree less by one than the degree of $f(x)$, with a fractional term due to the remainder. For example, let $f(x) = x^4 - 3x^3 - 7x + 2$ be divided by $x - 4$; the result is

$$\frac{x^4 - 3x^3 - 7x + 2}{x - 4} = x^3 + x^2 + 4x + 9 + \frac{38}{x - 4}$$

In the general case, we may write

$$\frac{f(x)}{x - b} = Q(x) + \frac{R}{x - b} \quad 14-2$$

where $Q(x)$ represents the new polynomial, and R is the remainder. If

* Any root of the equation $f(x) = 0$ is called a zero of the function.

now we multiply both members of this equation by $x - b$ to clear fractions, the result is

$$f(x) = (x - b)Q(x) + R$$

Now let us set $x = b$; we have

$$f(b) = (0)Q(b) + R$$

$$\therefore f(b) = R$$

as was to be proved.

If it happens that b is a zero of $f(x)$, then of course $f(b) = 0$. In that case the remainder vanishes. This proves the *factor theorem*: If the polynomial 14-1 has a zero r , then $x - r$ is a factor of $f(x)$.

Exercises

1. Use the factor theorem to prove that $x - a$ is a factor of $x^n - a^n$.
2. Prove that $x + a$ is a factor of $x^n - a^n$ only if n is even.
3. Show that $x + a$ is a factor of $x^n + a^n$ only if n is odd.
4. If $f(x) = 3x^4 - 7x^2 + 2x + 5$, find $f(3)$ by the remainder theorem, and check by direct substitution.

87. The Number of Roots. It may be proved by advanced methods that every polynomial equation has at least one root. This is called the fundamental theorem of algebra. Suppose that this root is r_1 . Then, by the factor theorem, we know that it is possible to divide out the root r_1 :

$$\frac{f(x)}{x - r_1} = Q(x)$$

The new polynomial $Q(x)$ must likewise have at least one zero, by the fundamental theorem of algebra. Let us call it r_2 . Proceeding in this way, we may write

$$f(x) = (x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n)a_0 \quad 14-3$$

There will be just n factors containing x , since the degree of the polynomial is diminished by one at each step. Hence *every polynomial equation of degree n has just n roots*.

88. Complex Roots. We next show that complex roots always occur paired. In equation 14-1, let us set

$$x = a + bi$$

After expanding and collecting the real and imaginary parts, we may write

$$f(a + bi) = A + Bi$$

Next, let us set

$$x = a - bi$$

Every term will be the same as in the previous operation, if i be replaced by $-i$. Hence

$$f(a - bi) = A - Bi$$

Now if $a + bi$ is a zero of the polynomial 14-1, we have

$$f(a + bi) = A + Bi = 0$$

Then $A = 0$ and $B = 0$, because the real and imaginary parts must vanish separately. Therefore

$$f(a - bi) = 0$$

This proves that, if $a + bi$ is a root of a polynomial equation, the complex conjugate $a - bi$ is also a root. Hence the total number of complex roots is an even number; the complex roots occur in pairs, the members of which are conjugate to each other.

89. Rational Roots. We next show that, if p/q is any rational zero of the polynomial 14-1, then

(a) p is a factor of the constant term a_n ; and

(b) q is a factor of the leading coefficient a_0 .

It is assumed that the fraction p/q has been reduced to lowest terms, so that p and q have no common factor greater than 1. By hypothesis,

$$a_0 \left(\frac{p}{q}\right)^n + a_1 \left(\frac{p}{q}\right)^{n-1} + \cdots + a_{n-1} \left(\frac{p}{q}\right) + a_n = 0 \quad 14-4$$

If both members of the equation be multiplied by q^{n-1} , the result may be written

$$\frac{a_0 p^n}{q} = -a_1 p^{n-1} - \cdots - a_{n-1} p q^{n-2} - a_n q^{n-1} \quad 14-5$$

Every term in the right-hand member is integral; hence $a_0 p^n / q$ is an integer. But by hypothesis, the rational number p/q is expressed in lowest terms; hence q must be a factor of a_0 .

Next, let every term in equation 14-5 be divided by p . The result may be written, after some transpositions,

$$\frac{a_0 p^{n-1}}{q} + 1 q^{n-2} = -a_1 p^{n-2} + \cdots + a_n \frac{q^{n-1}}{p}$$

Every term in the left-hand member is integral; hence $a_n q^{n-1} / p$ is an integer, and p must be a factor of a_n .

If information about the *nature* of some polynomial equation is desired, the foregoing discussion indicates that effective tools are available. Theoretical investigations often require just such information. Moreover, in some instances a complete solution may readily be attained with the help of these theorems.

Example. Consider the equation

$$f(x) = 3x^4 - x^3 + 3x^2 + 29x - 10 = 0$$

We first construct a table of possible rational roots, noting that $p = \pm 1, 2, 5$, or 10 ; and $q = 1$ or 3 . Hence any rational roots that may exist will be found in the table

$$\pm 1, 2, 5, 10$$

$$\pm \frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \frac{10}{3}$$

By trial, we find that -2 and $\frac{1}{3}$ are roots. Dividing them out, we have

$$f(x) = (x + 2)(3x - 1)(x^2 - 2x + 5)$$

The quadratic factor yields the remaining two roots, $1 + 2i$ and $1 - 2i$.

Exercises

Find the roots of the following equations.

1. $4x^3 + x^2 + x - 3 = 0$

Ans. $\frac{3}{4}, -\frac{1}{2} + \frac{\sqrt{3}i}{2}, -\frac{1}{2} - \frac{\sqrt{3}i}{2}$

2. $10x^3 - 29x^2 - 5x + 6 = 0$

3. $2x^4 + 7x^3 - 13x^2 + 7x - 15 = 0$

Ans. $\frac{3}{2}, -5, i, -i$

4. $2x^4 + x^3 - 7x^2 - 12x - 20 = 0$

5. $x^3 - 2x^2 - 13x + 6 = 0$

Ans. $-3, \frac{5 \pm \sqrt{17}}{2}$

6. Write the polynomial $14-1$ in the sigma notation.

90. Irrational Roots. Often an engineer is interested only in finding a particular root of a polynomial equation, as in the problem of the floating ball considered at the beginning of this chapter. The unknown root is usually irrational; its value may perhaps be approximately known by physical intuition, and the problem is to find the correct value to two, three, or four significant figures.

Let us suppose that a certain function has the value -2 when $x = a$, and $+3$ when $x = b$. We may further suppose that, as x changes from a to b , the function changes continuously from -2 to $+3$. This is possible

only if the function becomes zero at some intermediate point, as Figure 68 shows in graphic form. Thus a change of sign in $f(x)$ indicates the existence of a root of the equation $f(x) = 0$. This is true for any function which is

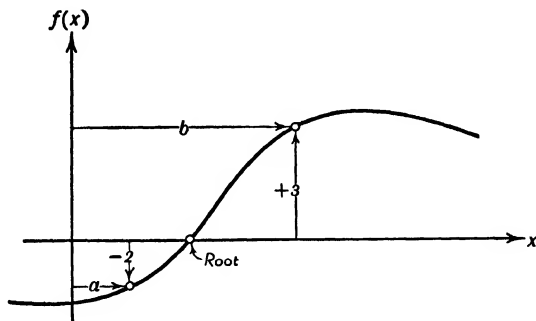


FIG. 68

continuous and single-valued in the interval in which the change of sign occurs; and all polynomial functions satisfy both of these mild conditions. Upon this theorem is based a method of attack which is simple in principle, and broad in scope.

Example 1. The equation $x^3 - x - 1 = 0$ has no rational roots. (The only possibilities are $+1$ and -1 .)

Let us form a table of values of

$$f(x) = x^3 - x - 1$$

x	$f(x)$
-2	-7
-1	-1
0	-1
+1	-1
+2	+5

The table reveals that a real root exists between $+1$ and $+2$. The formula for linear interpolation (9-13) suggests that $x = 1.2$ be tried. The result is

$$f(1.2) = -0.472$$

Hence $x = 1.2$ is too low. We now locate the root between successive tenths:

x	$f(x)$
1.3	-0.10
(?)	0.00
1.4	+0.34

As the interval is narrowed, the interpolation formula 9-13 grows more reliable:

$$0 = -0.10 + m(0.44)$$

$$m = 0.23$$

The new trial value is $x = 1.323$. We next locate the root between successive hundredths:

x	$f(x)$
1.32	-0.020
(?)	0.000
1.33	+0.023

Interpolating once more:

$$0 = -0.020 + m(0.043)$$

$$m = 0.46$$

Hence the root lies between 1.324 and 1.325.

Example 2. Find the depth to which a ball, whose specific gravity is 0.30, and whose radius is 2 inches, sinks in water.

Taking $R = 2$, and $\delta b = 0.3\delta w$, we have the equation

$$9.6 = h^2(6 - h)$$

We may set

$$f(h) = h^2(6 - h) - 9.6$$

The desired root is located between successive integers by the table

h	$f(h)$
0	-9.6
1	-4.6
2	+6.4

The next trial value is $h = 1.4$. The root is located between successive tenths by the table

h	$f(h)$
1.4	-0.58
(?)	0.00
1.5	+0.52

The nature of the data does not justify preparing a third table. The root is approximately $h = 1.45$.

91. The Method of Successive Approximations. The method of the preceding section, though general, is tedious; and a great amount of ingenuity has been expended in devising shorter methods. One of the most useful of these is set forth in the following examples.

Example 1. A problem in progressions leads to the equation

$$\frac{n}{2}(1+n) = 1500$$

In standard form, this becomes

$$n^2 + n - 3000 = 0$$

The following table shows that, in the neighborhood of the positive root, n is small as compared with n^2 or with 3000:

n	$f(n)$
50	-450
60	+660

If the equation is rewritten in the form

$$n^2 = 3000 - n$$

the last term may be considered as a (relatively) small correcting term. For a first approximation, let the last term be neglected. Then

$$n_1' = \sqrt{3000} = 54.77$$

Using this value for the correcting term, a better approximation to n_1 may be obtained.

$$n_1'' = \sqrt{2945.23} = 54.27$$

Employing n_1'' for the correcting term, a still better approximation for n_1 results:

$$n_1''' = \sqrt{2945.73} = 54.2745$$

It is clear that no further approximations are necessary for six-place work.

In this instance, one approximation gave a result correct to two places, the next approximation gave four places, and the third gave six places. Note that arithmetical work is automatically checked at each repetition of the process.

The sequence of numbers n_1' , n_1'' , n_1''' , \dots , converges on the first root n_1 . The second root is readily obtained from the relation

$$n_1 + n_2 = -1 \quad (\text{page 135})$$

$$\therefore n_2 = -55.2745$$

An independent check is afforded by calculating the product of the roots (page 135).

Example 2. The method just outlined is very flexible, but must be employed with some judgment. Thus the equation

$$5.8v^2 - 37.7v + 19.4 = 0$$

has two positive roots, one at about $v = \frac{1}{2}$, and the other at about $v = 6$, as a

table of values shows. In the case of the larger root, the constant term is small in comparison to the other terms. Hence if the equation is put in the form

$$v = \frac{37.7}{5.8} - \frac{19.4}{5.8v}$$

it may be readily solved by successive approximations.

For the smaller root, the second degree term is small in comparison to the others. The equation should therefore be put in the form

$$v = \frac{19.4}{37.7} + \frac{5.8v^2}{37.7}$$

for solution by successive approximations.

Example 3. The problem of the floating ball may be solved by successive approximations. Write the equation in the form

$$h^2 = \frac{9.6}{6 - h}$$

As a first approximation, neglect the quantity h in the right-hand member:

$$h' = \sqrt{\frac{9.6}{6}} = 1.3$$

Another approximation gives

$$h'' = \sqrt{\frac{9.6}{6 - 1.3}} = 1.43$$

The third approximation leads to

$$h''' = \sqrt{\frac{9.6}{6 - 1.43}} = 1.45$$

as before.

Exercises

1. Find the larger root of Example 2 by successive approximations, correct to two decimal places. Use the sum and product of the roots to complete and check the solution.
2. Find the smaller root of Example 2 by successive approximations, correct to two decimal places. Use the sum and product of the roots to complete and check the solution.
3. Find all of the roots of the equation

$$x^3 - 3x + 1 = 0$$

4. Find to two decimal places the positive root of

$$x^5 - 23x - 37 = 0$$

5. Find the positive root of the equation

$$x^3 + 32x - 145 = 0$$

correct to three significant figures.

Ans. 3.35

6. Find the height of a spherical segment, whose volume is 262 cubic inches, if the radius is 5 inches.

92. Trigonometric Equations. It is convenient to have a symbol for the angle whose sine, or whose tangent, is some quantity x . Accordingly, we write

$$A = \arcsin x \quad \text{if} \quad \sin A = x \quad 14-6$$

$$A = \arctan x \quad \text{if} \quad \tan A = x \quad 14-7$$

These equations define *inverse trigonometric functions*. The remaining trigonometric functions also possess inverses, but they are less frequently required in analysis. The symbols $\sin^{-1} x$ and $\tan^{-1} x$ are often used for the inverse sine and tangent functions. The significance of the exponent is operational, as is readily seen from the equation

$$\sin (\sin^{-1} x) = x$$

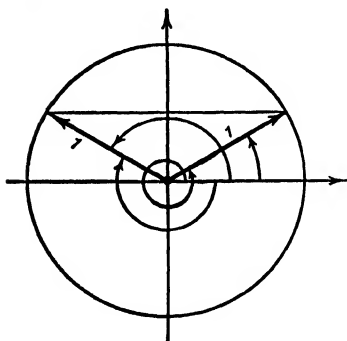


FIG. 69

An inverse trigonometric function may be regarded as constituting a formal solution to a trigonometric equation. Thus the equation $\sin A = \frac{1}{2}$ has the formal solution $A = \arcsin \frac{1}{2}$. We observe that an infinite number of angles satisfy the equation; four of them are shown in Figure 69. However, all of them may be obtained from either the first quadrant value or the second quadrant value by adding or subtracting some multiple of 360° . In the language of algebra,

$$A = 30 \pm n(360) \text{ degrees}$$

or

$$A = 150 \pm n(360) \text{ degrees} \quad (n = 1, 2, 3, \dots)$$

This simple example serves to indicate what we may expect to find when confronted with a trigonometric equation. There exists no such precise and well-organized body of theory as in the case of polynomial equations. Certain classes of trigonometric equations may be satisfactorily attacked by algebraic methods; if this proves to be impractical, numerical methods may be employed. With patience, the roots of any

trigonometric equation with numerical coefficients may be found by the method of section 90.

Example 1. It is required to find all angles between 0° and 360° that satisfy the equation

$$\sin A = 2 \cos A - 1$$

The first step is to obtain an equation in terms of a single trigonometric function. Both members are squared:

$$\sin^2 A = 4 \cos^2 A - 4 \cos A + 1$$

Then

$$1 - \cos^2 A = 4 \cos^2 A - 4 \cos A + 1$$

or

$$5 \cos^2 A - 4 \cos A = 0$$

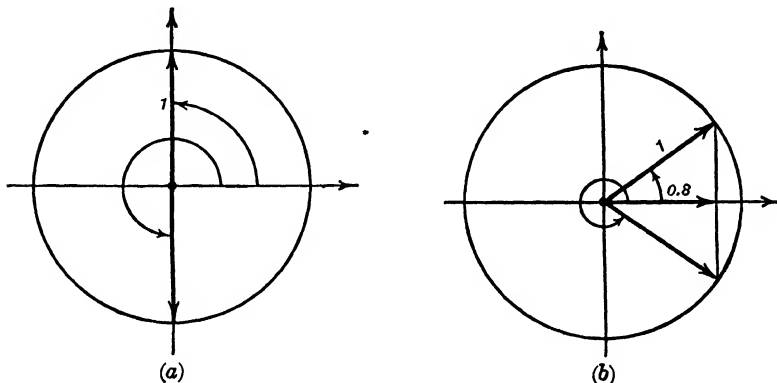


FIG. 70

This is a quadratic equation in $\cos A$; by the usual methods, we obtain

$$\cos A = 0 \quad \text{or} \quad 0.800$$

Hence

$$A = 36^\circ 50', 90^\circ, 270^\circ, \text{ or } 323^\circ 10'$$

Each of these roots must be checked separately. Even if no error has been made in the work, it is possible that some or all of the roots may not satisfy the original equation. In this example, it turns out that two do not check; they are *extraneous roots*, introduced by the operation of squaring.

$$\therefore A = 36^\circ 50' \quad \text{or} \quad 270^\circ$$

Example 2. Find the first-quadrant angles satisfying the equation

$$2 \sin 2x = \tan x$$

Using equation 10-12, we have

$$4 \sin x \cos x = \frac{\sin x}{\cos x}$$

Either $\cos x = \pm \frac{1}{4}$, or $\sin x = 0$. The negative answer may be neglected, because the cosine is not negative in the first quadrant. Hence

$$x = 0^\circ \text{ or } 60^\circ$$

Example 3. If B is an acute angle, find its value from the relation

$$\sin (25^\circ + B) = \cos (B - 14^\circ)$$

Using the complementary relations, the right-hand member may be transformed to $\sin (90^\circ + 14^\circ - B)$.

$$\sin (25^\circ + B) = \sin (104^\circ - B)$$

$$25^\circ + B = 104^\circ - B$$

$$\therefore B = 39^\circ 30'$$

Example 4. Find an angle in the first quadrant satisfying the equation

$$\frac{1}{\tan A - 2} = 2 \sin \frac{A}{2}$$

Evidently no first-quadrant solution exists unless $\tan A$ is greater than 2, so that only the interval from 63° to 90° need be considered. An inspection of the trigonometric tables shows that the left-hand member changes much more rapidly than does the right-hand member in this region. This is clearly brought out by the following table:

A	$\frac{1}{\tan A - 2}$	$2 \sin \frac{A}{2}$
66°	4.0	1.09
72°	0.93	1.18

It is therefore appropriate to use the method of successive approximations. Assume $A = 70^\circ$ in the right-hand member, and calculate its value from the resulting equation.

$$\frac{1}{\tan A' - 2} = 1.147 = \frac{1}{0.872}$$

$$\tan A' = 2.872$$

$$A' = 70^\circ 48'$$

Use this new value in the right-hand member:

$$2 \sin \frac{70^\circ 48'}{2} = 1.159 = \frac{1}{0.863}$$

Set

$$\frac{1}{\tan A'' - 2} = \frac{1}{0.863}$$

$$\therefore A'' = 70^\circ 45'$$

It is interesting to observe that if the assumed value of A be any acute angle at all the correct value will quickly be reached by the method above.

Exercises

Solve the following equations.

1. $5 \sin^2 x = \sin x + \cos^2 x$

Ans. $30^\circ \pm n(360^\circ)$
 $150^\circ \pm n(360^\circ)$
 $-19^\circ 28' \pm n(360^\circ)$
 $-160^\circ 32' \pm n(360^\circ)$

2. $6 \sin^2 x = 5 \cos x$

3. $A = 3 \tan A + 1$ (The angle is in radians.)

Ans.

Smallest positive root = 3.912

4. $2A = 3 \sin A - 2$ (The angle is in radians.)

5. $\sin x + 3 \cos x = 2$

Ans. $69^\circ 12' \pm n(360^\circ)$
 $-32^\circ 20' \pm n(360^\circ)$

6. $3 \sin \theta = 2 - 4 \cos \theta$

7. Solve Exercise 5 by making use of the trigonometric identity 10-25.

8. Solve Exercise 6 by making use of the identity 10-26.

9. A problem in mechanics leads to the simultaneous equations

$$\begin{cases} F_1 = 0.25 (50 \cos \theta) \\ F_2 = 0.25 (150 \cos \theta) \\ 100 \sin \theta = F_1 + F_2 \\ C = F_1 + 50 \sin \theta \end{cases}$$

Solve for θ and C .

10. The equation

$$m^2 - \left(A - \frac{1}{B} \right) m - 1 = 0$$

occurs in the theory of least squares.

(a) Show that $m = A$, approximately, if A and B are large numerically. Show that a better approximation is

$$m = A + \left(\frac{1}{A} - \frac{1}{B} \right)$$

(b) Show that $m = B$, approximately, if A and B are small numerically. A better approximation is

$$m = B + B^2(A - B)$$

(c) Show that one root of the quadratic equation always lies between A and B .

CHAPTER 15

PROGRESSIONS

93. Arithmetic Series. Expressions such as

$$2 + 5 + 8 + 11 + \dots$$

and

$$\frac{1}{4} - \frac{3}{4} - \frac{7}{4} - \frac{11}{4} - \dots$$

are called arithmetic series, or progressions. The general form is

$$S = a + (a + d) + (a + 2d) + \dots \quad 15-1$$

The letter a designates the first term of the progression. Each term, after the first, is formed from the preceding one by adding the quantity d to it; hence d is called the common difference.

The n th term is clearly

$$l = a + (n - 1)d \quad 15-2$$

We require a formula for the sum of n terms:

$$S_n = a + (a + d) + (a + 2d) + \dots + (l - d) + l$$

Reversing the order in which the terms are written, we have

$$S_n = l + (l - d) + (l - 2d) + \dots + (a + d) + a$$

Upon adding the corresponding terms of the right-hand members of these two equations, the d 's drop out, leaving n terms of the form $a + l$. Hence

$$2S_n = na + nl$$

or

$$S_n = n \left(\frac{a + l}{2} \right) \quad 15-3$$

As an aid to the memory, we observe that the sum of an arithmetic series equals the number of terms multiplied by the average of the first and last terms.

Equations 15-2 and 15-3 constitute two conditions upon the five quantities a , d , n , l , S_n . Hence if any three of these quantities are known, the other two may be found.

It is sometimes necessary to form an arithmetic progression, of which the first and last terms are known. This process is called *inserting arithmetic means* between the given numbers.

Note that the quantities a , $(a + l)/2$, l form an arithmetic progression, since

$$\frac{a + l}{2} - a = l - \frac{a + l}{2}$$

For this reason, the quantity $(a + l)/2$ is often called the arithmetic mean of a and l .

Example 1. Insert three arithmetic means between $+3$ and -5 .

This means finding an arithmetic progression containing five terms. We observe that $a = +3$, $l = -5$, and $n = 5$. Hence

$$-5 = +3 + (5 - 1)d$$

$$d = -2$$

The progression is $+3, +1, -1, -3, -5$.

Example 2. Find the sum of the first thousand even numbers. This sum is

$$S_n = 2 + 4 + 6 + \dots$$

The given quantities are $a = 2$, $d = 2$, and $n = 1000$.

$$l = 2 + (999)(2) = 2000$$

so that

$$S_n = 1000 \left(\frac{2 + 2000}{2} \right) = 1,001,000$$

Exercises

- Express in words the meaning of the three dots in equation 15-1.
- Express equation 15-1 in the sigma notation.
- Find the sum of the first n odd numbers. *Ans. n^2*
- How may a set of quantities be examined to determine whether they form an arithmetic progression? Apply your test to the following examples:
 - $2 + 9 + 15 + 20 + \dots$
 - $x + \frac{3x + y}{4} + \frac{x + y}{2} + \frac{x + 3y}{4} + \dots$
- Find the 5th term in the progression of Exercise 4(b) above.
- In a raffle, tickets marked 1, 2, 3, 4, and so on, are shaken up in a hat and drawn by purchasers one at a time. The price of a ticket is the number of cents corresponding to the number on the ticket. If the raffled article is worth \$15, what is the least number of tickets that will insure no loss to the person selling the tickets? *Ans. 55*

7. Derive the formula

$$d = \frac{2(S_n - an)}{n(n-1)}$$

8. Insert three arithmetic means between the numbers -2 and $+22$.

Ans. 4, 10, 16

9. Insert four arithmetic means between the numbers M and N .

94. Geometric Series. The expressions

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

and

$$6 - 0.6 + 0.06 - 0.006 + \dots$$

are called geometric series, or geometric progressions. The general form is

$$G = a + ar + ar^2 + \dots \quad 15-4$$

Each term, after the first, is formed from the preceding one by multiplying it by the constant ratio r .

The n th term is

$$l = ar^{n-1} \quad 15-5$$

The sum of the first n terms may be written

$$G_n = a + ar + ar^2 + \dots + ar^{n-1}$$

Multiplying both members by r :

$$rG_n = ar + ar^2 + ar^3 + \dots + ar^n$$

Subtracting each member of the second equation from the corresponding member of the first,

$$G_n - rG_n = a - ar^n$$

Hence

$$G_n = \frac{a(1-r^n)}{1-r} \quad 15-6$$

Thus far the theory of the geometric series parallels that of the arithmetic series. Equations 15-5 and 15-6 provide two conditions upon the five quantities a , r , n , l , G_n .

The process of forming a geometric series, when the first and last terms are known, is called *inserting geometric means* between the numbers.

The quantities a , $\sqrt[n]{al}$, l form a geometric progression, since

$$\frac{\sqrt[n]{al}}{a} = \frac{l}{\sqrt[n]{al}}$$

Hence the square root of the product of two quantities is called their geometric mean.

Example 1. Find the sum of 8 terms of the series

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$$

The constant ratio is $r = \frac{1}{3}$. The first term is $a = \frac{1}{3}$. Hence

$$\begin{aligned} G_8 &= \frac{\frac{1}{3}[1 - (\frac{1}{3})^8]}{1 - \frac{1}{3}} \\ &= 0.50 \end{aligned}$$

very nearly.

Example 2. A sheet of a translucent plastic 1 millimeter in thickness absorbs 80% of monochromatic light passing through it. How much light is absorbed by 8 millimeters of the plastic?

Each millimeter transmits 20% of the incident light; hence we seek the eighth term in the progression

$$\begin{aligned} 0.20, 0.20^2, 0.20^3, \dots \\ \therefore l = 0.20^8 \\ = 2.6(10^{-6}) \end{aligned}$$

That is, 0.000 26% is transmitted, and 99.999 74% is absorbed.

Exercises

- Express equation 15-4 in the sigma notation.
- How may a set of quantities be examined to determine whether they form a geometric series? Apply your test to the following examples:
 - $x^2 + xy + y^2$
 - 10, 100, 1000, ...
 - $\log 10, \log 100, \log 1000, \dots$
- Evaluate $\sum_{i=1}^{20} (1.06)^i$. *Ans.* 39.0
- Show that the sum of a geometric progression may be written in the alternative form

$$G_n = \frac{a(r^n - 1)}{r - 1}$$

For what values of r would this form be especially convenient?

- Show that the sum of n terms of the series

$$\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots$$

is given by the formula $\frac{p^n - 1}{p^n(p - 1)}$.

6. Find the sum of fifteen terms of the series

$$\frac{1}{1 - \frac{1}{2}} + \frac{\frac{1}{2}}{1 - \frac{1}{2}} + \frac{(\frac{1}{2})^2}{1 - \frac{1}{2}} + \dots$$

7. Obtain the formula

$$G_n = \frac{rl - a}{r - 1}$$

8. If each stroke of a pump removes 2% of the air in a sugar evaporator, how much air remains after 150 strokes? Ans. 5%
9. Insert two geometric means between -2 and 54 .
10. Insert three geometric means between M and N .

$$\text{Ans. } \sqrt[4]{M^3N}, \sqrt{MN}, \sqrt[4]{MN^3}$$

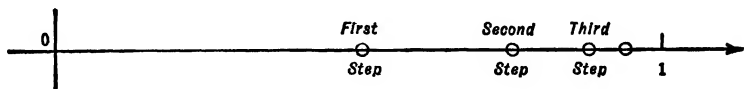


FIG. 71

95. The Series G_∞ . Suppose that we start at zero on the number scale, and make a step halfway to 1. Let the next step be half the remaining distance, and so on, each step diminishing the remaining distance by one-half (see Figure 71). We wish to consider the manner in which the point 1 is approached as the number of steps increases.

The situation may be described by the geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

It is clear that the sum increases as the number of terms increases; it is equally certain that the sum will never exceed 1, since each successive step takes us only halfway to that point. Here is a quantity which increases steadily and yet never exceeds a definite number. We say that such a quantity must *approach a limit*. In this case, it is geometrically evident that the limit approached by the sum of the series is 1.

This kind of situation is one whose importance in pure and applied mathematics cannot be overestimated. For this reason, we shall discuss at some length the key theorem:

The infinite geometric series 15-4 converges to the limit $a/(1 - r)$ when r is less in absolute value than 1. There is no limit if r is equal to or greater than 1.

To prove the first part of the theorem, it is necessary to show that, when r is less than 1 in absolute value, the quantity r^n is very small when the

exponent n is very large. In more precise language, r^n approaches the limiting value zero as n increases without limit.

We shall begin by considering a quantity of the form

$$(1 + \Delta)^n$$

where Δ is any positive number, known in advance; and n is a positive integer. We observe that

$$(1 + \Delta)^2 = 1 + 2\Delta + \Delta^2$$

$$(1 + \Delta)^3 = 1 + 3\Delta + 3\Delta^2 + \Delta^3$$

and so on. It appears that

$$(1 + \Delta)^n = 1 + n\Delta + \dots$$

where the dots represent terms whose exact representation need not concern us; they are all, however, positive. (This formula may be rigorously established by the binomial theorem. See Chapter 22.)

We conclude that $(1 + \Delta)^n$ is a larger number than $(1 + n\Delta)$.

The next step is to examine the behavior of the quantity $(1 + n\Delta)$ as n increases indefinitely. Suppose that $n = 1, 2, 3, \dots$. It is clear that a value of n can be found, beyond which $(1 + n\Delta)$ is larger than any preassigned number, N , however large N may be. This rather complicated idea is briefly expressed, in symbols, thus:

$$(1 + n\Delta) \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty$$

Another symbolism for the same idea is

$$\lim_{n \rightarrow \infty} (1 + n\Delta) = \infty$$

But if $(1 + n\Delta)$ increases indefinitely with increasing n , so must $(1 + \Delta)^n$. Upon setting $1 + \Delta = x$, we see that x^n increases without limit, with increasing n , if x is greater than 1.

We are now prepared to consider the behavior of r^n , if r is a positive number less than 1. Set

$$r = \frac{1}{x}$$

$$r^n = \frac{1}{x^n}$$

If r is less than 1, then x must be greater than 1. Now as n becomes larger and larger, x^n becomes larger than any preassigned number, N ,

however large. It follows that r^n becomes smaller than $1/N$, which may be made arbitrarily small. Hence r^n approaches zero as a limit, with increasing n , if r is less in absolute value than 1. Symbolically,

$$\lim_{n \rightarrow \infty} r^n = 0 \quad \text{for } r^2 < 1$$

This means that r^n becomes and remains as close to 0 as we desire, if n be taken large enough.

The rest of the proof is easy. The quantity

$$\frac{a}{1-r}$$

is a constant independent of n . Hence we can choose such a value of n that

$$\frac{ar^n}{1-r}$$

becomes arbitrarily small. Finally, since

$$G_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

we have

$$G_\infty = \frac{a}{1-r} \tag{15-7}$$

as was to be proved.

The second half of the theorem presents no difficulties. It is easy to see that

$$a + a + a + a + \dots$$

and

$$a - a + a - a + \dots$$

do not converge. The first grows by equal steps beyond any finite number. The second skips back and forth between 0 and a . Hence the series diverges for $r = \pm 1$.

If r is greater numerically than 1, the succeeding terms in the progression 15-4 become larger and larger. This establishes the second part of the theorem.

Example 1. Express the repeating decimal $1.1818\dots$ as a rational number.

$$1.1818\dots = 1 + (0.18 + 0.0018 + \dots)$$

The quantity in parentheses is an infinite geometric series. It is evident that

$a = 0.18$ and $r = 0.01$. Hence

$$\begin{aligned} 1.1818 \dots &= 1 + \frac{0.18}{1 - 0.01} \\ &= 1 + \frac{18}{11} \\ &= 1 + \frac{1^2}{11} \\ &= \frac{12}{11} \end{aligned}$$

Example 2. Theoretically, when a ball having a coefficient of elasticity of 0.8 is dropped upon a hard surface, it rebounds to a height equal to eight-tenths of the height from which it was dropped. Suppose it is released from a height of 15 feet, and continues to rebound. How far does it travel before coming to rest?

$$\begin{aligned} x &= 15 + 2(0.8)(15) + 2(0.8^2)(15) + \dots \\ &= 15 + 2 \left(\frac{15}{1 - 0.8} \right) \\ &= 15 + \frac{30}{0.2} \\ &= 165 \text{ feet} \end{aligned}$$

Of course, the physical ball does not rebound an infinite number of times; but equation 15-7 gives a good approximation, and is much simpler to employ than 15-6.

Exercises

1. Taking $r = \frac{1}{2}$, find a value of n such that

$$r^n < 0.001$$

2. Taking $r = 0.99$, find a value of n such that

$$r^n < 10^{-6}$$

3. Express the repeating decimal $2.616161 \dots$ as a fraction in lowest terms.
 4. Express the repeating decimal $0.7135135 \dots$ as a fraction in lowest terms.
 5. Find the limits approached by the following series:

(a) $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots$

(b) $18 - 9 + \frac{9}{2} - \frac{9}{4} + \dots$

6. Show that the logarithms of the successive terms in a geometric series form an arithmetic series.

96. Compound Interest. Suppose that a sum of money P bears interest at the rate of 3% per annum. At the end of one year, each dollar has become \$1.03. The total amount is $P(1.03)$.

If the interest rate is i , at the end of one year the amount of principal plus interest is

$$A_1 = P(1 + i)$$

At the end of the year, the total amount equals the sum invested at the beginning of the year multiplied by $(1 + i)$. Hence at the end of the second year the amount of principal plus interest is

$$A_2 = A_1(1 + i) = P(1 + i)^2$$

After n years, the amount of principal plus interest is

$$A_n = P(1 + i)^n \quad 15-8$$

This is the compound interest formula. When interest is compounded more often than once a year, the same formula is valid, except that n now represents the *number of interest periods*, and i the *rate of interest per interest period*.

Suppose, for example, that the sum of \$200 bears interest for 15 years, at the rate of 4% compounded semi-annually. The number of interest periods is 30; the rate of interest per interest period is 2%. Hence the amount is

$$200(1 + 0.02)^{30}$$

97. Investment and Annuity Problems. An interesting application of the theory of progressions is found in certain problems of finance.

Example 1. A man makes annual deposits of \$200, starting now, in a bank which pays 4% compounded semi-annually. How much will he have to his credit at the end of 15 years?

The 1st deposit bears interest for 15 years and amounts to	$200(1.02)^{30}$
The 2nd deposit bears interest for 14 years and amounts to	$200(1.02)^{28}$
...	...
The last deposit bears interest for 1 year and amounts to	$200(1.02)^2$

Writing these in reverse order, we have

$$\begin{aligned} G_n &= 200(1.02)^2 + 200(1.02)^4 + \cdots + 200(1.02)^{30} \\ &= 200(1.02)^2 \left[\frac{(1.02)^{30} - 1}{(1.02)^2 - 1} \right] \end{aligned}$$

Errors in setting up problems of this kind are not infrequently due to confusion

caused by the fact that the letter n is used for two distinct quantities: the number of interest periods in formula 15-8; and the number of terms in the progression in formula 15-6. Before employing the latter formula, it is well to verify the correctness of the progression by using equation 15-5.

Example 2. What sum must be set aside from the yearly earnings of a manufacturing company to replace a \$50,000 building at the end of 30 years, if money is worth 5% compounded annually?

Let x be the required annual sum.

The 1st deposit (made at the end of the 1st year) bears interest for 29 years and amounts to	$x(1.05)^{29}$
The 2nd deposit bears interest for 28 years and amounts to	$x(1.05)^{28}$
...	...
The last deposit (which has no time to bear interest) amounts to	x

The progression contains 30 terms. Taking x as the first term, we have

$$50,000 = x \left[\frac{(1.05)^{30} - 1}{1.05 - 1} \right]$$

$$\therefore x = \frac{2500}{(1.05)^{30} - 1}$$

Example 3. If we make 10 annual deposits of \$50, starting now, in a bank paying $3\frac{1}{2}\%$ compounded annually, how much will have accumulated after 25 years?

The 1st deposit bears interest for 25 years and amounts to	$50(1.035)^{25}$
The 2nd deposit bears interest for 24 years and amounts to	$50(1.035)^{24}$
...	...
The 10th deposit bears interest for 16 years and amounts to	$50(1.035)^{16}$

Reversing the order of terms in the progression as before, we have

$$G_n = 50(1.035)^{16} \left[\frac{(1.035)^{10} - 1}{1.035 - 1} \right]$$

In computations performed with the aid of ordinary five-place tables of logarithms, it will be found that the final result is often good to only three or four significant figures. For greater precision, seven-place tables of logarithms may be employed. The use of interest tables, especially devised for calculations of this kind, reduces the labor of computation materially.

Exercises

1. Show that for example 1, on page 176, $G_n = \$4178.91$. (Make a complete outline before beginning logarithmic computation.)
2. What annual premium, payable at the beginning of each year, would provide for a life insurance policy of \$5000 maturing 38 years hence, assuming that interest is compounded annually at the rate of 3%?
3. If we make 20 annual deposits of \$100 beginning now and are allowed 5% interest compounded annually, how much will there be to our credit 20 years from now?
Ans. \$3471.93
4. If we make 20 annual deposits of \$100 beginning now and are allowed 4% interest compounded annually, how much will there be to our credit at the end of 25 years?
5. How much must a father deposit annually to establish a fund of \$1000 for sending his boy to college at the age of 18, if he begins when the boy is 5 years old, and interest is at 3% compounded semi-annually? (Include final deposit, made on eighteenth birthday.)
Ans. \$58.44
6. How much must be deposited at the beginning of each year, for 12 years only, in a bank paying 2% interest, compounded quarterly, to amount to \$1200 at the end of 15 years?

CHAPTER 16

ANALYTIC GEOMETRY: THE STRAIGHT LINE

98. Cartesian Coordinates. We have on several occasions found it useful to represent a function

$$y = f(x)$$

by a graph. It is time for us to undertake a closer examination of the relationship existing between algebra and geometry.

To every pair of real numbers

$$(x_1, y_1)$$

there corresponds a point on a plane (see Figure 72). Conversely, to every point there corresponds a pair of numbers. The first number is the *abscissa*, or distance to the y -axis. The second number is the *ordinate*, or distance to the x -axis.

99. The Slope Formula. Consider the line segment PQ connecting two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. PQ is of course a line vector (see Figure 73); we proceed to express some familiar properties of vectors in a new form, suited to the present purpose.

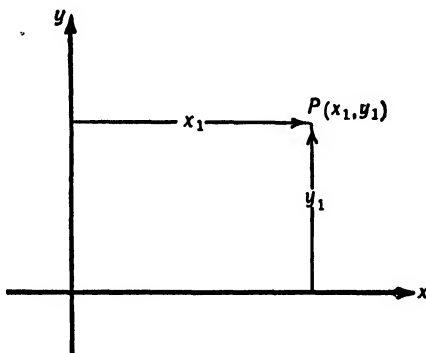


FIG. 72

The **horizontal component** of the vector PQ is the difference of the abscissas, $x_2 - x_1$.

The **vertical component** of PQ is the difference of the ordinates, $y_2 - y_1$.

16-1

The direction of PQ is specified by the *slope*, which is the tangent of the angle made by PQ with the direction of the positive x -axis. Hence the

slope is given by the vertical component divided by the horizontal component. That is,

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad 16-2$$

Figure 72 illustrates the simplest of various possible cases, namely, that in which all coordinates are positive, and the components of PQ are also positive. Formula 16-1 is, however, valid for any two points P and Q ;

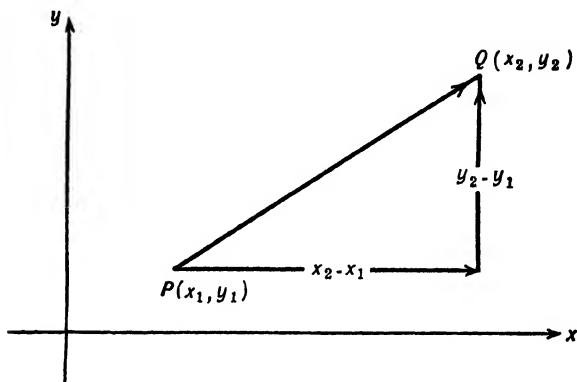


FIG. 73

and formula 16-2 is valid except when $x_2 = x_1$. Observe that the horizontal component of PQ is negative when Q lies to the left of P , and that the vertical component is negative when Q lies below P . The components of the vector running from Q to P are equal numerically to the components of PQ , but are opposite in sign. The slopes of PQ and QP are, however, equal.

100. The Distance Between Two Points. If d is the length of the vector PQ , the theorem of Pythagoras may be expressed in the form

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad 16-3$$

The *distance formula* 16-3 and the *slope formula* 16-2 express the most fundamental properties of a line vector, namely direction and magnitude, in terms of the coordinates of its endpoints. It will be found that for two particular points it is usually easier to substitute in the formulas, instead of working directly from a figure.

Example. The points $A(-2,2)$, $B(0,-4)$, and $C(10,6)$ are the vertices of a triangle. Let us find the lengths and slopes of the three sides.

$$AB = \sqrt{(0+2)^2 + (-4-2)^2} = \sqrt{40}$$

$$BC = \sqrt{(10-0)^2 + (6+4)^2} = \sqrt{200}$$

$$CA = \sqrt{(10+2)^2 + (6-2)^2} = \sqrt{160}$$

Since $\overline{BC}^2 = \overline{AB}^2 + \overline{CA}^2$, the triangle is a right triangle.

$$m_{AB} = \frac{-4-2}{0+2} = -3$$

$$m_{BC} = \frac{6+4}{10-0} = +1$$

$$m_{CA} = \frac{6-2}{10+2} = +\frac{1}{3}$$

The quantities $(0+2)$, $(-4-2)$, and so forth, which occur in these formulas represent the projections of the sides of the triangle on the coordinate axes. All

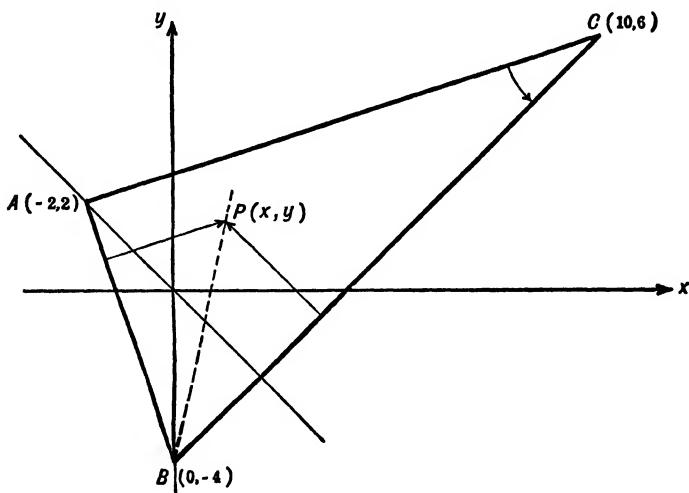


FIG. 74

of the foregoing results may be checked graphically (see Figure 74). It should be observed that the slopes of CA and BC are *positive*, so that the lines go up to the right. The slope of AB is *negative*, so that the line goes down to the right. Moreover, the slope of BC is greater than the slope of CA .

Exercises

These exercises (and subsequent ones) should be solved analytically, and checked from the figure.

1. In which quadrants are the signs of x and y alike? In which quadrants unlike?
2. Express formula 16-2 in words.
3. Find the magnitude, slope, and components of the line vectors running from
(a) $(2,7)$ to $(-4,3)$. (b) $(5,-2)$ to $(-6,-6)$.
4. Find the lengths and slopes of the sides of the triangles whose vertices are
(a) $(1,6)$, $(-2,5)$, $(-1,-4)$. (b) $(-2,-3)$, $(4,-1)$, $(3,-8)$.
5. Show that the points $(-2,-3)$, $(2,0)$, $(-1,-5)$, and $(1,2)$ are vertices of a parallelogram.
6. Show that the points $(10,-2)$, $(6,2)$, and $(4,0)$ are vertices of a right triangle.
7. Draw the triangle whose vertices are $A(0,3)$, $B(-6,3)$, and $C(0,-2)$. If A and C are opposite vertices of a parallelogram, find the point D which is the vertex opposite B .
8. If $ABCD$ represents any quadrilateral, show that the horizontal component of AD is equal to the algebraic sum of the horizontal components of AB , BC , and CD .

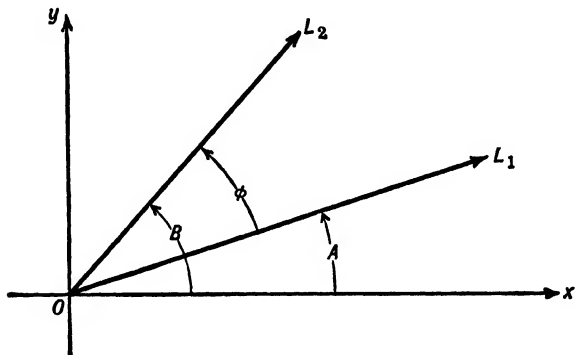


FIG. 75

101. The Angle Between Two Lines. It is sometimes convenient to express the angle between two lines or line vectors in terms of their slopes. Let m_1 be the slope of OL_1 , and m_2 the slope of OL_2 (Figure 75). Then

$$\begin{aligned}\tan \phi &= \tan (B - A) \\ &= \frac{\tan B - \tan A}{1 + \tan A \tan B}\end{aligned}$$

from equation 10-7. Hence

$$\tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Example. Find angle C in the right triangle discussed on page 181.

The angle is positive if measured counterclockwise, from CA to CB . An arrow should be drawn to represent the sense of the angle, as in Figure 74. In using equation 16-4, the quantity m_2 is taken to be the slope of the line to which the head of the arrow is pointing. Thus

$$\tan C = \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{1}{2}$$

$$\therefore C = 26^\circ 34'$$

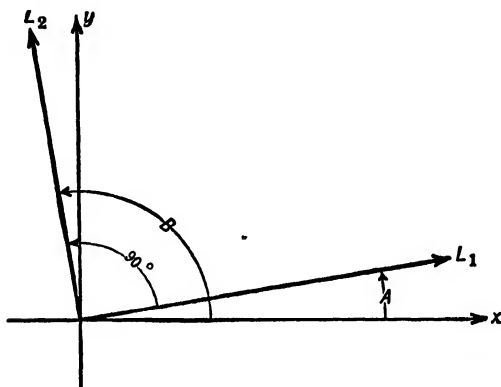


FIG. 76

102. Perpendicular Lines. In the special case $\phi = 90^\circ$, we have (Figure 76):

$$\begin{aligned} m_2 &= \tan B \\ &= \tan (90^\circ + A) \\ &= -\cot A && \text{(page 107)} \\ &= -\frac{1}{\tan A} \\ m_2 &= -\frac{1}{m_1} && \text{16-5} \end{aligned}$$

That is, the slopes of perpendicular lines are negative reciprocals of each other.

Example. The slope of AB (Figure 74) is -3 ; the slope of CA is $+\frac{1}{3}$. Since these values satisfy equation 16-5, the lines are perpendicular.

Exercises

1. Show that the triangle whose vertices are $(3,7)$, $(9,-5)$, and $(0,-2)$ is a right triangle.
2. Find the smallest angle of the triangle whose vertices are $(2,-3)$, $(3,1)$, $(-4,2)$.
Ans. $31^\circ 40'$
3. Find the largest angle of the triangle whose vertices are $(-4,-3)$, $(1,1)$, $(3,0)$.
4. Find the slope of the altitude upon the shortest side of the triangle of Exercise 2.
Ans. $-\frac{1}{4}$
5. Find the slope of the altitude upon the longest side of the triangle of Exercise 3.
6. Find all of the angles of the triangle whose vertices are at the points $(-2,4)$, $(3,3)$, and $(2,-5)$.
Ans. $54^\circ 43'$, $94^\circ 12'$, $31^\circ 5'$

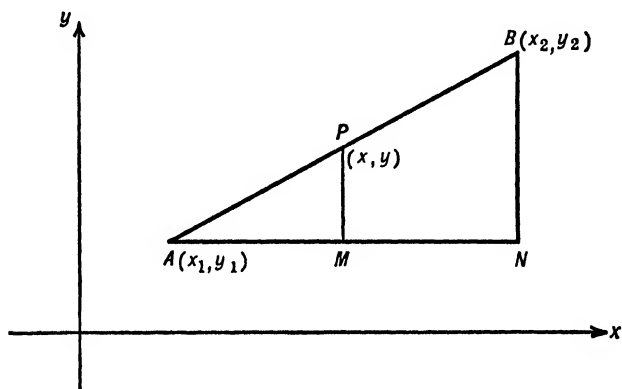


FIG. 77

103. The Midpoint Formula. When the coordinates of two points are known, the coordinates of the midpoint of the line segment joining them may be obtained as follows.

Let $P(x, y)$ be the midpoint of AB (Figure 77). We wish to express x and y in terms of x_1 , y_1 , x_2 , and y_2 . Since $AP = \frac{1}{2}AB$, the horizontal components are in the same ratio:

$$AM = \frac{1}{2}AN$$

$$x - x_1 = \frac{1}{2}(x_2 - x_1)$$

$$x = \frac{x_1 + x_2}{2} \quad 16-6$$

In like manner,

$$y = \frac{y_1 + y_2}{2} \quad 16-7$$

The discussion is readily generalized. Suppose that the segment AP is equal in length to AB multiplied by a certain fraction r . Then

$$AM = r(AN)$$

$$x - x_1 = r(x_2 - x_1)$$

$$\therefore x = x_1 + r(x_2 - x_1) \quad 16-8$$

In like manner, we find

$$y = y_1 + r(y_2 - y_1) \quad 16-9$$

Exercises

- Find a point one-third of the way from $A(1, -2)$ to $B(-5, -8)$. (In analytic geometry, to "find a point" means to find the coordinates of the point.)
Ans. $(-1, -4)$
- Find the point which lies between $(-3, 1)$ and $(5, 7)$, and which is three times as far from the first point as from the second.
- Given $A(2, 6)$ and $B(1, -3)$, find a point along the line passing through A and B which is twice as far from A as from B .
Ans. $(\frac{4}{3}, 0), (0, -12)$
- Find the points of trisection of the line segment running from $(-2, -6)$ to $(4, 2)$.
Ans. $(0, -\frac{10}{3}), (2, -\frac{2}{3})$
- The points $A(-3, 4)$, $B(1, -4)$, and $C(5, 6)$ determine a triangle.
 - Find the midpoint P of side BC .
 - Find a point one-third of the way from P to A .
 - Find the midpoint R of side AB .
 - Find a point one-third of the way from R to C .
 - Show that the medians of this triangle intersect at a point of trisection of each.
- By the method of Exercise 5, find the point at which the medians of a triangle whose vertices are (x_1, y_1) , (x_2, y_2) , (x_3, y_3) intersect each other.

$$\text{Ans. } \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

- Show that the midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

104. The Straight Line. A line is completely specified if one point on it, and the direction, are known. Let $A(x_1, y_1)$ be the known point. The direction is determined if the slope m is known. Suppose that $P(x, y)$ represents any point on the line (Figure 77). We may apply the slope formula 16-2 to the segment AP :

$$m = \frac{y - y_1}{x - x_1}$$

or

$$y - y_1 = m(x - x_1) \quad 16-10$$

This is called the point-slope form of the equation of a line.

From the manner in which equation 16-10 is obtained, it is clear that the equation is satisfied by the coordinates of every point on the line, and by no other point. Hence we say that the equation represents the line.

Equations 16-2 and 16-10 are very similar in form. The essential difference lies in the fact that the coordinates of the point $P(x,y)$ are not known in advance, as is indicated by the absence of a subscript.

If the line is parallel to the x -axis, equation 16-10 reduces to

$$y = y_1 \quad 16-11$$

If the line is parallel to the y -axis, the foregoing treatment is not valid; however, it is readily seen that the form

$$x = x_1 \quad 16-12$$

represents the line in this case.

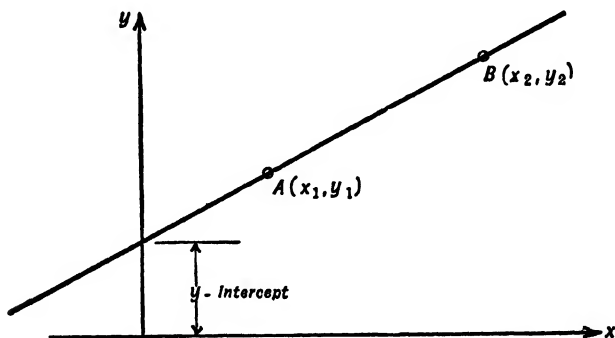


FIG. 78

105. The Slope-Intercept Form. If equation 16-10 is solved for y , the result is

$$y = mx + y_1 - mx_1$$

The expression $y_1 - mx_1$ can be represented by a single constant, say b . Thus we arrive at the form

$$y = mx + b \quad 16-13$$

This is the linear function 6-1. We may interpret the coefficient of x geometrically: it is the *slope* of the line. The constant b represents the ordinate of the line at $x = 0$: it is the *y-intercept* of the line (see Figure 78).

Exercises

1. Draw on one set of coordinate axes the lines

$$y = 2x + 7$$

$$y = -\frac{x}{2} + 7$$

$$y = -3x + 7$$

$$y = \frac{x}{4} + 7$$

What geometric meaning may be attached to the constant 7 in each case?

2. As in Exercise 1, for the lines

$$y = 2x + 6$$

$$y = 2x + 1$$

$$y = 2x - 2$$

$$y = 2x - 5$$

What geometrical statement may be made concerning all lines of the form $y = 2x + k$?

3. What geometric object is represented by the equation $x = 0$? By the equation $y = 0$?
4. Find the equation of the median, upon side AB , of the triangle whose vertices are $A(2,6)$, $B(-8,0)$, and $C(4,-2)$. *Ans.* $5x + 7y - 6 = 0$
5. Find the equation of the altitude upon side AB of the triangle of Exercise 4. *Ans.* $5x + 3y - 14 = 0$
6. Find the equation of the line which is perpendicular to side AB at its midpoint, for the triangle of Exercise 4. *Ans.* $5x + 3y + 6 = 0$
7. Show that the medians, the altitudes, and the perpendicular bisectors of the triangle of Exercise 4 are concurrent, and that the points of concurrency are collinear.

106. The General Linear Form. The equation

$$Ax + By + C = 0$$

16-14

is the most symmetrical form for the first degree function. It is evidently the same as equation 16-13, apart from a constant factor. Upon solving for y , we observe that the slope of the line is $-A/B$. Hence the direction of the line is that of a vector whose horizontal component is B , and whose vertical component is $-A$.

Since the slope of a line perpendicular to the line 16-14 is $+(B/A)$, the coefficients A and B may be taken to be the horizontal and vertical com-

ponents of a vector *normal* (that is, perpendicular) to the line 16-14 (see Figure 79).

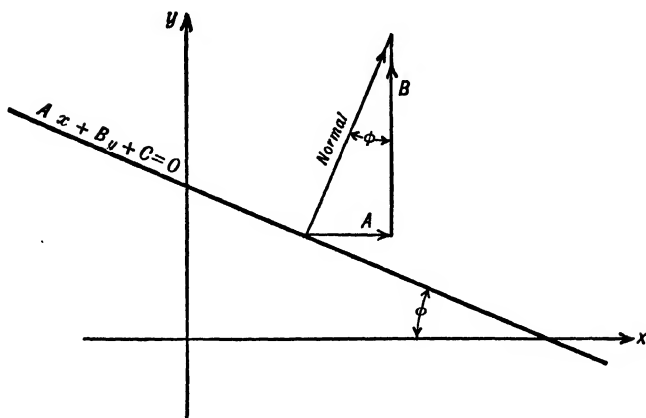


FIG. 79

In Figure 79, the angle ϕ made by the line with the horizontal is equal to the angle made by the normal with the vertical. This angle is such that

$$\tan \phi = \frac{A}{B}$$

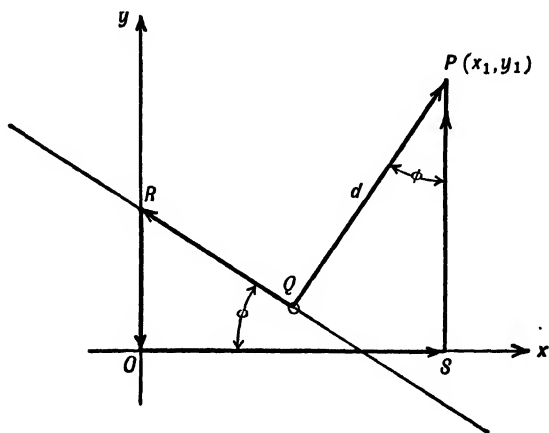


FIG. 80

107. The Distance from a Point to a Line. Referring to Figure 80, let $d = QP$ be the vector normal from the line $Ax + By + C = 0$ to the

point $P(x_1, y_1)$. QP may be regarded as the vector sum of QR , RO , OS , and SP . Taking components of these five vectors in a direction normal to the line, we find

$$\text{Normal component of } QP = d$$

$$\cdot \text{ Normal component of } QR = 0$$

$$\text{Normal component of } RO = RO \cos \phi$$

$$\text{Normal component of } OS = OS \sin \phi$$

$$\text{Normal component of } SP = SP \cos \phi$$

Moreover, we easily find (Figures 79 and 80)

$$RO = \frac{C}{B} \quad \cos \phi = \frac{B}{\sqrt{A^2 + B^2}}$$

$$OS = x_1$$

$$SP = y_1 \quad \sin \phi = \frac{A}{\sqrt{A^2 + B^2}}$$

Since QP is the vector sum of the other four line vectors, its component in any direction equals the algebraic sum of the components of the other four vectors. Hence

$$d = 0 + \frac{C}{B} \left(\frac{B}{\sqrt{A^2 + B^2}} \right) + x_1 \left(\frac{A}{\sqrt{A^2 + B^2}} \right) + y_1 \left(\frac{B}{\sqrt{A^2 + B^2}} \right)$$

which may be written

$$d = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \quad \text{16-15}$$

It will be found that formula 16-15 always gives a positive result for points lying on one side of the line, and a negative result for points lying on the other side. The reason for this will be evident when the equation of the plane is discussed (Chapter 20).

From equation 16-15, it is easily seen that the distance from the *origin* to the line $Ax + By + C = 0$ is, apart from sign,

$$\frac{C}{\sqrt{A^2 + B^2}}$$

The sign is determined by the sign of the constant term C . Hence, for all points on the same side of the line as the origin, the sign of d in formula

16-15 will be the same as that of C . For all points on the opposite side from the origin, the sign will be opposite to that of C .

If $C = 0$, the line passes through the origin. In this case, any convenient point may be selected to test formula 16-15 for sign.

Example. Find the equation of the straight line which bisects angle B , in the triangle formed by the points $A(-2,2)$, $B(0,-4)$, and $C(10,6)$ (see Figure 74).

The first step is to find the equation of the line passing through points A and B . The slope of this line is -3 , and its y -intercept is -4 . From equation 16-13, we have for the equation of the line AB

$$y = -3x - 4$$

or, in the general form,

$$3x + y + 4 = 0$$

In the same way, the equation of the line passing through points B and C is found to be

$$x - y - 4 = 0$$

Now let $P(x,y)$ be any point on the angle bisector. From elementary geometry, we know that every point on the bisector of an angle is equidistant from the two sides. The distance of P from the side AB is, apart from sign,

$$d_1 = \frac{3x + y + 4}{\sqrt{3^2 + 1^2}}$$

The distance of P from the side BC is, apart from sign,

$$d_2 = \frac{x - y - 4}{\sqrt{1^2 + 1^2}}$$

These distances are equal numerically; but d_1 and d_2 have opposite signs. If, for example, point P is inside the triangle, the sign of d_1 is positive, as determined by the rule given above. But d_2 is negative when P lies within the triangle. Hence the coordinates of P must satisfy the equation

$$d_1 = -d_2$$

which reduces to

$$y = (2 + \sqrt{5})x - 4$$

Exercises

1. Show that the point $(2, -3)$ lies midway between the lines $4x - 2y = 0$ and $2x + y + 6 = 0$.
2. Find the area of the triangle with vertices at $(-3,0)$, $(9,-2)$, and $(2,2)$ by finding the length of the longest side, and the altitude upon it. As a check, calculate the area by means of equation 11-4.
3. Find the bisectors of the angles between the lines $x + 2y = 4$ and $2x - y = 2$.
4. Find the equation of the line bisecting the largest angle of the triangle of Exercise 2.

108. Straight Line Determined by Two Conditions. A particular straight line may be specified by means of two geometric conditions; for example, by two points lying on the line, or by one point and the slope. It will be recalled (Chapter 6) that the term "condition" implies that an algebraic equation can be written, expressing the condition. Hence two conditions imply that two equations can be found, in which the constants of the straight line formulas occur as unknowns. There must, then, be just two essential arbitrary constants in any straight line equation. This is clearly brought out in the slope-intercept form 16-13.

The general form 16-14 appears at first sight to contain three arbitrary constants. That only two of these are independent may be shown as follows: Divide both members by A ; the result is

$$x + \frac{B}{A}y + \frac{C}{A} = 0$$

or

$$x + B'y + C' = 0$$

This last form contains just two arbitrary constants.

The point-slope form 16-10 has been shown to contain only two independent constants.

Example. Let us find the equation of the line passing through point $A(-2,2)$, and perpendicular to a line passing through $B(0,-4)$ and $C(10,6)$. The first condition leads to the equation

$$2 = m(-2) + b$$

For, since the point A lies on the line, its coordinates must satisfy the equation of the line (taken in the slope-intercept form). The second condition leads to

$$m = -1$$

since the slope of the required line is the negative reciprocal of the slope of the line segment BC , which is readily found by the slope formula. The two conditional equations in the unknowns m and b are solved, and the equation of the required line is found to be

$$y = -x$$

This is the equation of a line passing through the origin of axes, and running down to the right at an angle of 45° . The result is in agreement with the diagram (Figure 74).

Exercises

1. Given two points $A(-6,3)$ and $B(2,-5)$:
 - (a) Find the equation of the line passing through A and B .
 - (b) Find the equation of a line perpendicular to AB at its midpoint.

2. Show that the condition for parallelism of two lines

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

is that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

3. Given the triangle $A(3,1)$, $B(-1,2)$, $C(2,-4)$:

- Find the equation of the line passing through A and B .
- Find the equation of the line passing through C , perpendicular to AB .
- Find the length of AB , and of the altitude upon AB .
- Calculate the area. Check by finding an angle and another side of the triangle, using a trigonometric formula for the area.

4. Prove that the area of any triangle is given by the formula

$$\text{Area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) are the vertices, taken in counterclockwise order.

Miscellaneous Problems

- Prove analytically that the diagonals of any parallelogram bisect each other.
- Show that the line segment joining the midpoints of the non-parallel sides of the trapezoid with vertices $(-7,7)$, $(1,-5)$, $(5,3)$, and $(1,9)$ is parallel to the bases of the trapezoid and equal to half their sum.
- Find the length and slope of the line segment joining the points $A(-5,4)$ and $B(4,-2)$. What are the coordinates of the point in which the line passing through A and B intersects a line parallel to the y -axis, and three units to the left of it?
- Write the equation of the perpendicular bisector of the line segment AB in Problem 3.
- Find the distance from the point $(-1,7)$ to the line passing through the points $(-5,4)$ and $(4,-2)$.
- Find the largest angle of the triangle whose vertices are the points $(6,1)$, $(2,8)$, and $(-4,6)$.
- Find the point of intersection of the altitudes of the triangle in Problem 6, and prove that all three altitudes pass through the same point.
- Find the equation of the line which passes through the point $(-5,3)$ and is parallel to the line $x - 4y = 6$.
- Show that the points $(-17,11)$, $(-10,-6)$, $(8,6)$, and $(0,18)$ all lie on a circle with center at $(-5,6)$.
- A man deposits \$45 at the end of each year, for 10 years only, in a bank which pays 3% interest compounded semi-annually. How much will he have to his credit at the end of 18 years?

11. Show that the midpoint of the hypotenuse of any right triangle is equidistant from the three vertices. [Suggestion: Let the vertices be the points $(b,0)$, $(0,a)$, and $(0,0)$.]
12. The compound interest formula is often written in the form

$$A = P \left(1 + \frac{i}{q} \right)^{qn}$$

where n represents the number of years, and interest is at the rate i per annum, compounded q times per annum. Solve for i in terms of the other letters.

13. If $f(x)$ is the general polynomial function 14-1, show that $f(-x)$ may be obtained from $f(x)$ by changing the sign of every term of odd degree.
14. Find the real part and the imaginary part of the complex expressions $\frac{1}{a + bi}$ and

$$\frac{a - bi}{a + bi}.$$

15. A delivery chute for boxes is so designed that the terminal velocity of the boxes will not exceed a certain limiting value. For one such chute, the angle of inclination is calculated from the equation

$$W \sin \theta - \frac{1}{5} W \cos \theta = \frac{2W}{5g}$$

Find θ , taking $g = 32.2$.

16. From the simultaneous equations

$$\begin{cases} D - N \sin A + fN \cos A = 0 \\ W - N \cos A - fN \sin A = 0 \end{cases}$$

eliminate the angle A by squaring and adding, and obtain the relation

$$D^2 + W^2 = N^2(1 + f^2)$$

(These equations occur in the study of axle friction.)

17. A thickness of 10 centimeters of flint glass transmits 88.0% of blue-green radiation incident upon it. Find the % transmitted by 0.5 centimeters of glass. (Suggestion: Construct a geometric series of 20 terms for the radiation transmitted by 0.5, 1.0, 1.5, \dots , centimeters of glass. The last term of the series is known.)

18. An alternating e.m.f. is applied to an electrical circuit whose impedance is unknown, connected in series with a pure resistance (Figure 81). The e.m.f. across the combination, 120 volts, is the vector sum of E_r , the voltage across the resistance, and E_z , the voltage across the impedance. If E_r is 22 volts, and E_z is 108 volts, what is the angle between the vectors E_r and E_z ? (The cosine of this angle is the *power factor* of the unknown impedance, an important quantity in applications.)

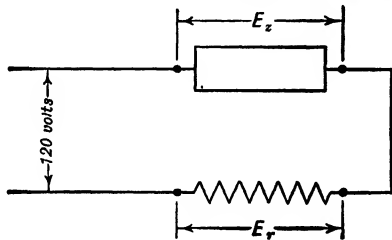


FIG. 81

19. If two direct-current circuits are connected in parallel, the total current which flows is the sum of the currents in the two branches. If alternating current is used, the total current is the *vector sum* of the currents in the two branches. If the currents in the branches are $I_1 = 5.11$ amperes and $I_2 = 4.02$ amperes, and if the total current is found to be $I = 7.25$ amperes, find the angle (called the "difference in phase") between each pair of current vectors (Figure 82).

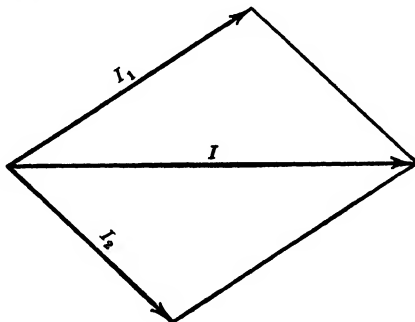


FIG. 82

20. Show that the conjugate of the product of two complex numbers is equal to the product of their conjugates. (Suggestion: Take the complex numbers to be $a + bi$ and $c + di$.)
21. In order to find the natural vibration frequencies of an elastic rod with a weight fastened to one end, it is necessary to solve the equation

$$a = \beta \tan \beta$$

in which the angle β is expressed in radians. Find the two smallest roots of this equation when $a = 1.08$.

CHAPTER 17

THE CONICS

109. Introduction. It was discovered by the Greek geometers that any curve, resulting from the intersection of a plane with a cone, is either an ellipse, a parabola, a hyperbola, or a limiting form of one of these curves. (A circle is regarded as a special case of an ellipse, for reasons which will appear.) The discovery of this theorem may be regarded as the culmination of the great age of Greek mathematics. Considering the mathematical methods then available, it was a colossal achievement, although in the light of present knowledge its importance must be judged to be minor.

All curves obtained in this manner are called conic sections, or conics. They are next in simplicity to those geometric objects, constructed from points and lines, which were studied in the previous chapter. With each of the conics there is associated a particular algebraic equation, by means of which the geometric properties of the conic may be investigated. These equations are important in their own right. For the student of applied mathematics, the insight that will be gained into the meaning and manipulation of these equations is a valuable by-product of the study of the conics.

110. Equation of the Circle. Let us find an equation to represent a circle of radius r with center at the origin of axes (Figure 83). It can be seen from the figure that the equation

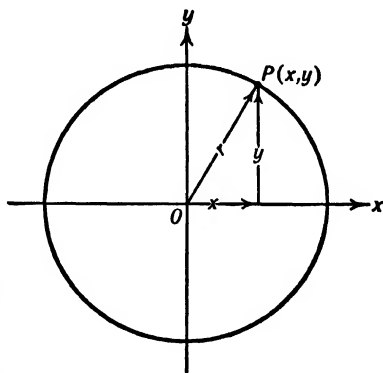


FIG. 83

$$x^2 + y^2 = r^2$$

17-1

is satisfied by every point $P(x, y)$ on the circle, and by no other point. Hence equation 17-1 represents the circle.

Next, consider the circle of radius r , with center at the point (h, k) .

From Figure 84 we see that the equation

$$(x - h)^2 + (y - k)^2 = r^2 \quad 17-2$$

is satisfied by every point $P(x,y)$ on the circle, and by no other point. Hence equation 17-2 represents any circle in the xy plane.

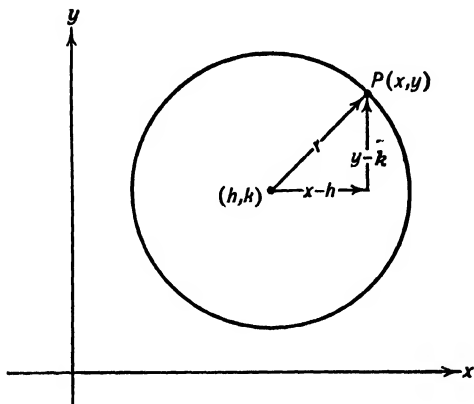


FIG. 84

It is evident that equation 17-2 is formally equivalent to the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad 16-3$$

From this point of view, equation 17-2 states that the distance from the center (h,k) to any point $P(x,y)$ on the circle is equal to a constant r . This is of course the most familiar property of a circle. The use of x and y , instead of x_2 and y_2 , implies that equation 17-2 refers to *any point on the circle*, and not to some particular point, known in advance.

111. Circle Determined by Three Conditions. The equation of a circle may be written in a form that is algebraically simpler than equation 17-2. Upon expanding, we have

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$

When the terms are rearranged, the result is

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

Expressing the constant term, and also the coefficients of x and y , by a

single letter, the last equation may be written

$$x^2 + y^2 + Dx + Ey + F = 0 \quad 17-3$$

Both of the forms 17-2 and 17-3 contain three essential constants; hence, three geometric conditions are required to specify a circle.

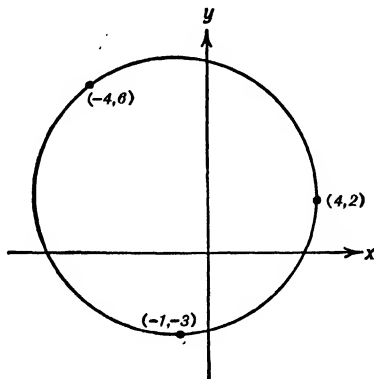


FIG. 85

Example 1. Find the equation of the circle passing through the three points $(-1, -3)$, $(4, 2)$, and $(-4, 6)$.

Since the point $(-1, -3)$ lies on the circle, its coordinates must satisfy the equation of the circle. Let us take the latter in the form 17-3, where D , E , and F are to be determined. Then

$$(-1)^2 + (-3)^2 + D(-1) + E(-3) + F = 0$$

Using the remaining two conditions, we have

$$(4)^2 + (2)^2 + D(4) + E(2) + F = 0$$

$$(-4)^2 + (6)^2 + D(-4) + E(6) + F = 0$$

The geometric conditions have now been expressed by three equations in the unknowns D , E , and F . Upon solving simultaneously, we find that

$$D = 2 \qquad E = -4 \qquad F = -20$$

The equation of the circle is therefore

$$x^2 + y^2 + 2x - 4y - 20 = 0$$

Another method of solution employs the principle of the well known method for construction of a circle passing through three given points. The equation of the line, which is perpendicular to the line segment joining the two points

$(-1, -3)$ and $(4, 2)$ at its midpoint, is found as in Chapter 16. The equation of this perpendicular bisector is

$$x + y - 1 = 0$$

The equation of the perpendicular bisector of the line segment joining a second pair of points $(-1, -3)$ and $(-4, 6)$ is found in the same way to be

$$x - 3y + 7 = 0$$

The center of the circle is the point of intersection of the two perpendicular bisectors. Solving simultaneously, we find that the center of the circle is $(-1, 2)$. The radius is now calculated by means of the distance formula 16-3, and is found to be 5. Hence the equation of the circle is

$$(x + 1)^2 + (y - 2)^2 = 25$$

Example 2. Find the equation of the smaller of the two circles, which are tangent to the y -axis and to the line $L_1: 3x + 4y - 18 = 0$, and whose centers lie on the x -axis.

Let (h, k) be the center of a circle satisfying the three given conditions. The distances from (h, k) to the y -axis and to the line L_1 are equal. From equation 16-15, we have

$$\pm h = \frac{3h + 4k - 18}{\sqrt{3^2 + 4^2}}$$

Since the center of the circle lies on the x -axis, we see that $k = 0$. Hence the two possible centers are $(-9, 0)$ and $(\frac{9}{4}, 0)$. Since the circle is tangent to the y -axis, the radius is numerically equal to h . Hence the center of the smaller circle is $(\frac{9}{4}, 0)$, and its radius is $\frac{9}{4}$. The equation is

$$(x - \frac{9}{4})^2 + y^2 = (\frac{9}{4})^2$$

which reduces to

$$2x^2 + 2y^2 = 9x$$

Exercises

1. Draw on one pair of coordinate axes the circles represented by the equations
 - (a) $(x - 2)^2 + (y - 3)^2 = 25$
 - (b) $(x - 2)^2 + (y - 3)^2 = 1$
 - (c) $(x - 2)^2 + (y - 3)^2 = 42$
2. If in equation 17-3, the constant term is missing (that is, $C = 0$), what geometric meaning can be attached to this fact?
3. The ends of a diameter of a circle are the points $(1, 9)$ and $(9, 3)$. Find the equation of the circle
 - (a) by locating the center and finding the radius;
 - (b) by using the fact that the square of the diameter equals the sum of the squares of the distances from a point $P(x, y)$ on the circle to the given points; and
 - (c) by using the fact that the two lines joining $P(x, y)$ to the given points are perpendicular.

4. Find the equation of the larger of the two circles satisfying the conditions of example 2 in the text. Verify graphically that both circles fulfill the three conditions.
5. Find the equation of the circle which passes through the points $(4, -4)$, $(5, 1)$, and $(-1, -3)$. *Ans.* $x^2 + y^2 - 4x + 2y - 8 = 0$
6. Find in expanded form the equation of a circle with center on the x -axis, and having the radius a , if the circle passes through the origin. *Ans.* $x^2 + y^2 = 2ax$
7. Find the equation of a circle with center on the y -axis, and having the radius a , if the circle passes through the origin.
8. Find the equation of a circle with center on the line $y = x$, and having the radius $\sqrt{2}a$, if the circle passes through the origin. *Ans.* $x^2 + y^2 = 2ax + 2ay$
9. Find the equations of two circles tangent to the x -axis, passing through the points $(0, 5)$ and $(2, 1)$.
10. Find the equations of the circles whose centers lie on the line $y = 2x + 4$, which are tangent to the line $y = x + 1$, and which pass through the point $(0, 3)$.
11. Find the equations of the two circles which have their centers on the line $x + 3y + 1 = 0$, and which are tangent to the two lines $2x - y + 5 = 0$ and $x + 2y + 6 = 0$. *Ans.* $45x^2 + 45y^2 + 30y - 251 = 0$
 $5x^2 + 5y^2 + 40x - 10y + 69 = 0$

112. Equation of the Parabola. We have discussed, by analytic methods, the properties of straight lines and circles. We now turn to less familiar geometric objects.

The parabola may be defined as a curve all of whose points are equidistant from a fixed point and a fixed line. The fixed point is called the **focus**.

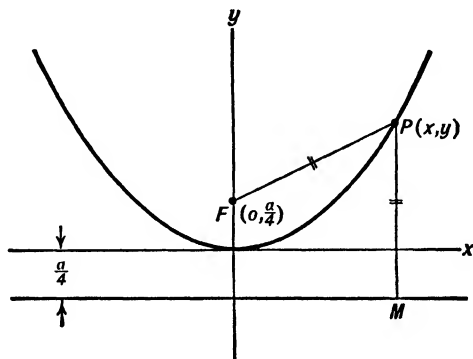


FIG. 86

The fixed line is called the **directrix**. In order that the equation of the curve may be obtained in a simple form, let us assume that the coordinates of the focus are $(0, a/4)$, and that the directrix is parallel to the x -axis, at a distance of $a/4$ units below it.

Referring to Figure 86, if $P(x, y)$ is any point on the parabola, we have

by definition

$$FP = PM$$

With the help of the distance formula 16-3, we may write our equation in the form

$$\sqrt{(x - 0)^2 + \left(y - \frac{a}{4}\right)^2} = y + \frac{a}{4}$$

Since this equation is satisfied by the coordinates of every point on the curve, and by no other point, it represents the parabola. Upon squaring both members, and simplifying, we obtain the equation of the parabola in the form

$$x^2 = ay \quad 17-4$$

The y -axis is an *axis of symmetry*. Let us see how the symmetry of the curve about this axis is revealed by its equation.

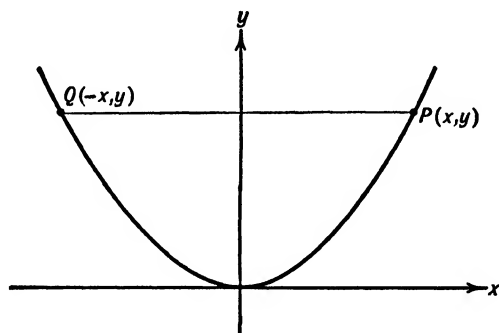


FIG. 87

The two points P and Q (Figure 87) are said to be symmetrical with respect to the y -axis. We may visualize one point as the mirror image of the other, the position of the mirror being indicated by the y -axis. The ordinates of P and Q are equal. The abscissas are equal numerically, but opposite in sign. Hence a curve may be tested for symmetry about the y -axis by substituting $(-x)$ for (x) in the equation of the curve. If the equation is unaltered, the curve is symmetrical about the y -axis.

Applying the test to equation 17-4, we have

$$(-x)^2 = x^2 = ay$$

This proves that the y -axis is an axis of symmetry.

An important geometric property of the parabola, which can be deduced directly from equation 17-4, may be stated as follows: The distance from any point on a parabola to the line which is tangent at the vertex is proportional to the square of the distance to the axis of symmetry.

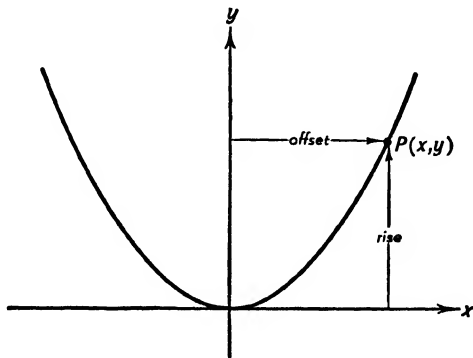


FIG. 88

tional to the square of the distance to the axis of symmetry. If the parabola is placed as in Figure 88, we may say that *the rise is proportional to the square of the offset*.

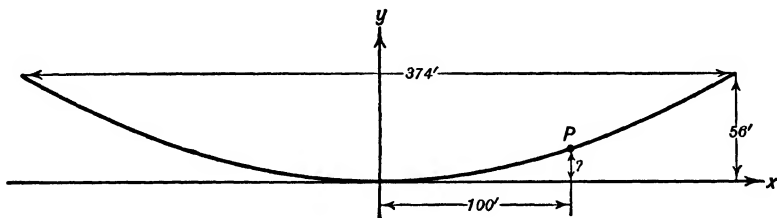


FIG. 89

Example 1. The cable of a suspension bridge follows the shape of a parabola. The span is 374 feet, and the sag is 56 feet (Figure 89). What is the height (measured from the lowest point) of the cable at a distance of 100 feet from the center?

Since the rise is proportional to the square of the offset, we may write

$$\frac{y}{100^2} = \frac{56}{187^2}$$

The proportion may be conveniently solved by slide rule: Set the indicator to 56 on the A scale; move the slide until 187 on the C scale lies under the hair-

line; opposite the index (100) read the answer (16 ft.) on the A scale.

A	56	(?)
C	187	100

The problem may also be attacked by finding the equation of the curve. The constant a in equation 17-4 may be determined by using the fact that the point (187, 56) lies on the curve. Hence

$$187^2 = a(56)$$

$$\therefore a = 624$$

The equation is

$$x^2 = 624y$$

Setting $x = 100$, we find as before

$$y = \frac{100^2}{624} = 16 \text{ feet}$$

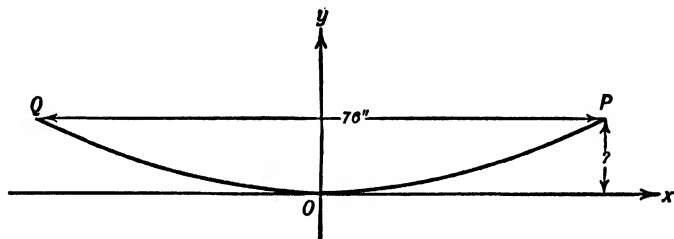


FIG. 90

Example 2. The reflecting mirror in an astronomical telescope is to have a focal length of 44 feet, and is to be 76 inches in diameter. Find the depth to which a glass block must be ground down at the center to make the mirror.

A sketch of the cross-section of the mirror is shown in Figure 90. The curve QOP must be ground to the shape of a parabola. The focal length (distance from the center of the mirror to the focal point) is

$$\frac{a}{4} = 44 \text{ feet} = 528 \text{ inches}$$

Hence the equation of the parabola is

$$x^2 = 2110y$$

When $x = 38$,

$$\begin{aligned} y &= \frac{38^2}{2110} \\ &= 0.685 \text{ inches} \end{aligned}$$

This is the required depth of the mirror at the center.

Exercises

1. If P is a point whose coordinates are $(2,7)$, find the point Q_1 which is the mirror image of P in the y -axis. Find also the point Q_2 which is an image point with respect to the x -axis.
2. Obtain an analytic test for symmetry of a curve with respect to the x -axis.
3. Find the equation of the parabola which passes through the points $(-3,7)$, $(0,0)$, and $(3,7)$.
4. Draw, on a single pair of coordinate axes, the parabolas whose equations are

$$x^2 = 4y$$

$$x^2 = y$$

$$x^2 = \frac{y}{4}$$

Locate the focus of each curve.

5. The block of glass, from which has been made the mirror for the most powerful telescope thus far constructed, measures 200 inches in diameter. The focal length of the mirror is 666 inches. Find the depth of the mirror at the center.
6. Show that the quantity a in equation 17-4 gives the length of a chord of the parabola, passing through the focus, and parallel to the directrix. (This chord is called the latus rectum.)
7. The cables of the George Washington bridge are supported by twin towers 3500 feet apart. The middle of the parabolic arcs formed by these cables is 325 feet below the ends. Find an equation which represents one of these cables. Also find the rise (distance above the middle point) of points at distances of 100, 200, and 400 feet from the middle.

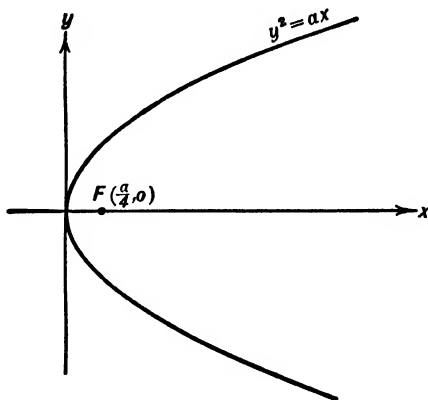


FIG. 91

113. Parabola with Horizontal Axis. The equation of a parabola in the position shown in Figure 91 can be deduced by observing that in the new position, the quantities x and y are merely interchanged, so that the equation of the curve becomes

$$y^2 = ax$$

If the constant a is negative, x must also be negative if y is to be real. Hence the parabola lies entirely in the second and third quadrants; that is, it opens to the left (Figure 92). A similar statement may be made re-

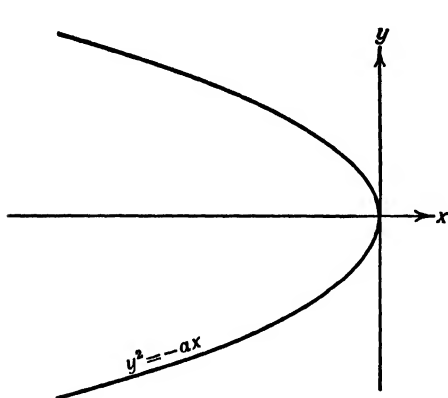


FIG. 92

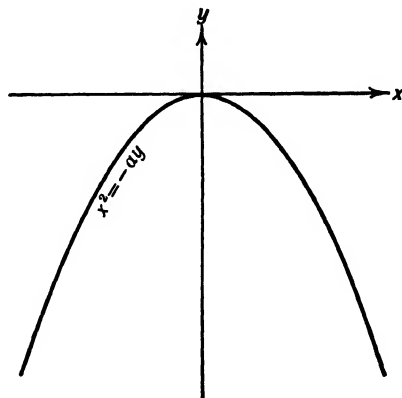


FIG. 93

garding equation 17-4; if the constant a is negative, the parabola opens downward (Figure 93).

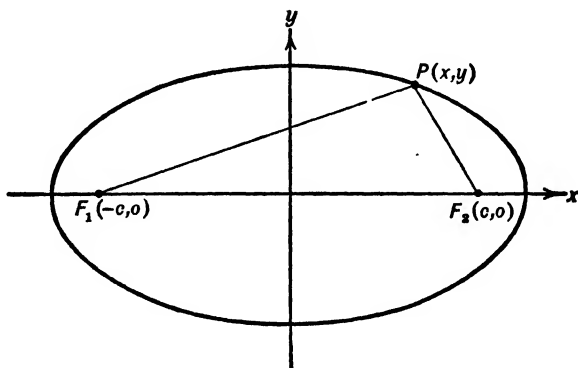


FIG. 94

114. The Ellipse. The ellipse may be defined as a curve such that the sum of the distances from any point on the ellipse to two fixed points is constant. For example, in Figure 94,

$$F_1P + PF_2 = 2a$$

for any point $P(x, y)$ on the ellipse. The equation will be obtained in its simplest form if we select $F_1(-c, 0)$ and $F_2(c, 0)$ as the two fixed points, and $2a$ as the constant sum of the distances F_1P and F_2P . The distance formula gives us immediately

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

This equation is satisfied by every point on the ellipse, and by no other point. It therefore represents the ellipse. In order to put it in a simpler form, we write

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

Upon squaring both members and simplifying, we arrive at

$$a\sqrt{(x-c)^2 + y^2} = a^2 - cx$$

Squaring once more, the result is

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

which readily reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

Finally, we set

$$b^2 = a^2 - c^2 \quad 17-6$$

The equation of the ellipse now assumes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad 17-7$$

The intercepts of the ellipse are exhibited by the table

x	y
0	$\pm b$
$\pm a$	0

The length of the horizontal diameter is evidently $2a$; that of the vertical diameter is $2b$.

From equation 17-6 we see that a is greater than b . This results from the initial assumption that the x -axis passes through the two foci, F_1 and F_2 . Had we assumed that the focal points were on the y -axis, and that the sum of the distances was $2b$, we should have found

$$a^2 = b^2 - c^2 \quad 17-8$$

instead of equation 17-6. (See Figure 95, and Exercise 2 below.)

If $a = b$, equation 17-7 may be written

$$x^2 + y^2 = a^2$$

which is a circle. A circle may therefore be regarded as a special kind of ellipse.

Example. An arch in the shape of a semi-ellipse has a span of 80 feet, and a height at the center of 22 feet. Find the height at a distance of 20 feet from the center.

Taking the origin of axes at the center of the ellipse, the equation is seen to be

$$\frac{x^2}{40^2} + \frac{y^2}{22^2} = 1$$

When $x = 20$, we have

$$\left(\frac{20}{40}\right)^2 + \left(\frac{y}{22}\right)^2 = 1$$

$$\left(\frac{y}{22}\right)^2 = 1 - \frac{1}{4}$$

$$\frac{y}{22} = \frac{\sqrt{3}}{2}$$

$$y = 11\sqrt{3} \\ = 19.1$$

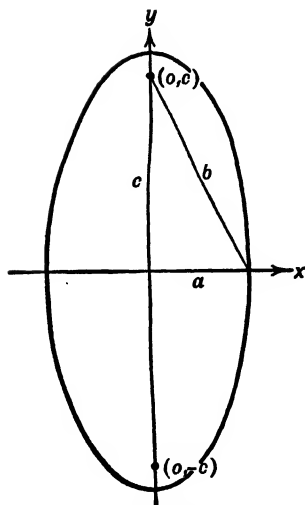


FIG. 95

to the nearest tenth of a foot.

In drawing an ellipse, additional points (other than the intercepts) should be located in the manner indicated by the foregoing example, so that the shape of the ellipse may be truer than when drawn by eye. See also Exercises 7 and 8 below.

Exercises

1. Draw on a single pair of axes the curves whose equations are

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

2. Derive the equation of an ellipse whose major diameter is vertical (see Figure 95). Take $(0, c)$ and $(0, -c)$ as the fixed points, and let the sum of the distances from any point on the ellipse to these fixed points be $2b$.

3. High bridges are sometimes supported by an arch in the shape of a parabola with vertical axis, opening downward. The level roadway of a bridge of this kind is 80 feet above the ends of the parabolic arch, and 10 feet above the middle point. The span of the bridge is 300 feet. Find the height of the roadway above the arch at a point 50 feet from one end.
4. Draw the parabolas

$$x^2 + 20y = 0$$

$$x + 4y^2 = 0$$

5. In bridges of moderate size, such as masonry bridges for modern highways, an arch in the form of a semi-ellipse is often used. A certain bridge of this type has a span of 70 feet, and a height of 20 feet at the center. Find the heights at distances of 15 and 30 feet from one end.
6. Eliminate the angle ϕ from the simultaneous equations

$$\begin{cases} x = a \cos \phi \\ y = b \sin \phi \end{cases}$$

and show that the resulting equation represents an ellipse.

7. One end, R , of a rod is free to slide in a groove along the positive x -axis. The other end, S , slides in a groove along the positive y -axis. Point $P(x, y)$ on the rod is at a distance a from S , and b from R . Show that

$$\begin{cases} x = a \cos \phi \\ y = b \sin \phi \end{cases}$$

where ϕ is the angle ORS . What curve is described by the point P ? Explain how one edge of a strip of stiff paper can be used to plot points for drawing an ellipse with semi-axes of length 6 and 9.

8. Two circles of radius a and b are drawn, concentric about the origin of axes, O . A line ON is drawn to a point N on the first circle, cutting the second at a point M . A vertical line drawn through N intersects a horizontal line through M in the point $P(x, y)$. Show that

$$\begin{cases} x = a \cos \phi \\ y = b \sin \phi \end{cases}$$

where ϕ is the angle XON . Hence show that the point P lies on an ellipse whose semi-axes are a and b . (This method for laying out an ellipse is preferred by draftsmen.)

9. Show that equation 17-7 represents a curve that is symmetrical about both of the coordinate axes.

115. The Hyperbola. We next consider the curve (hyperbola) whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad 17-9$$

If x is replaced by $(-x)$, equation 17-9 is unaltered; hence the hyperbola is symmetrical about the y -axis. Similar reasoning shows that it is symmetrical about the x -axis. We need therefore consider at first only that

part of the curve lying in the first quadrant; the rest can be obtained by considerations of symmetry.

Upon solving for y , the equation of the hyperbola assumes the form

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2} \quad 17-10$$

The curve becomes imaginary in the region

$$-a < x < a$$

When $x = a$, $y = 0$. As x increases, y increases also, rapidly at first, and then more slowly. In order to understand the nature of the curve for large values of x , we write equation 17-10 in the form

$$y = \pm \frac{bx}{a} \sqrt{1 - \frac{a^2}{x^2}} \quad 17-11$$

When x is large in comparison to a , the fraction a^2/x^2 becomes negligibly small, and the equation 17-11 approaches the limiting form

$$y = \pm \frac{bx}{a} \quad 17-12$$

The last equations represent a pair of straight lines, called the asymptotes of the hyperbola. The hyperbola is said to approach these lines asymptotically as x becomes indefinitely larger.

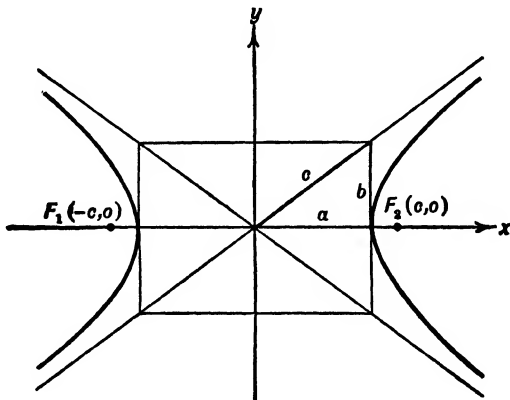


FIG. 96

The foregoing results are exhibited in Figure 96. In order to sketch a hyperbola, a rectangle of height $2b$ and width $2a$ is drawn about the origin

of axes. The diagonal lines through the corners of this rectangle are the asymptotes. The two branches of the hyperbola lie between the asymptotes, and are tangent to the sides of the rectangle. The x -axis is in this instance called the *axis of the hyperbola*.

The points $F_1(-c,0)$ and $F_2(c,0)$, where

$$c^2 = a^2 + b^2 \quad 17-13$$

are called the foci of the hyperbola. These points play a role in the theory of the hyperbola, analogous to that of the corresponding points in the theory of the ellipse and of the parabola.

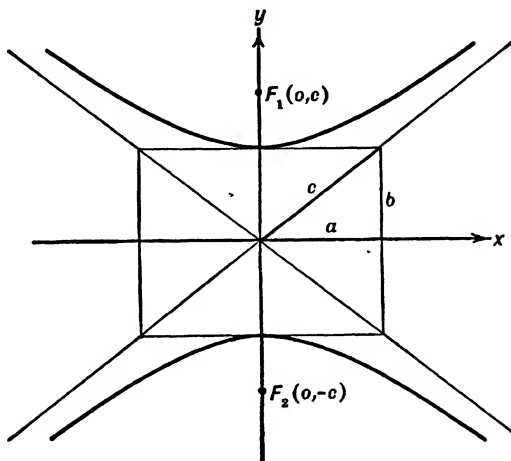


FIG. 97

If the focal points lie on the y -axis, the equation of the hyperbola assumes the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \quad 17-14$$

The curve corresponding to equation 17-14 is shown in Figure 97. The y -axis is, in this case, the axis of the hyperbola. The x -axis is called the conjugate axis.

Exercises

1. Draw on a single pair of coordinate axes the hyperbolas

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \qquad \frac{y^2}{9} - \frac{x^2}{16} = 1$$

Locate the foci of both hyperbolas.

2. Derive the equation of the curve for which the *difference* of the distances from any point $P(x,y)$ on the curve to the points $(5,0)$ and $(-5,0)$ is constantly 8 units.
3. Derive the equation of the curve for which the difference of the distances from any point $P(x,y)$ on the curve to the points $(c,0)$ and $(-c,0)$ is constantly $2a$ units.
4. Find the equation of the hyperbola whose axis is the x -axis, whose asymptotes are the lines $3x = \pm 5y$, and whose x -intercepts are ± 10 .
5. Draw the hyperbola

$$4y^2 - 16x^2 = 16$$

6. (a) Show that the equations of the asymptotes of the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2$$

are

$$bx \pm ay = 0$$

- (b) Hence show that the product of the distances from any point on the hyperbola to the asymptotes is a constant, whose value is

$$\frac{a^2b^2}{a^2 + b^2}$$

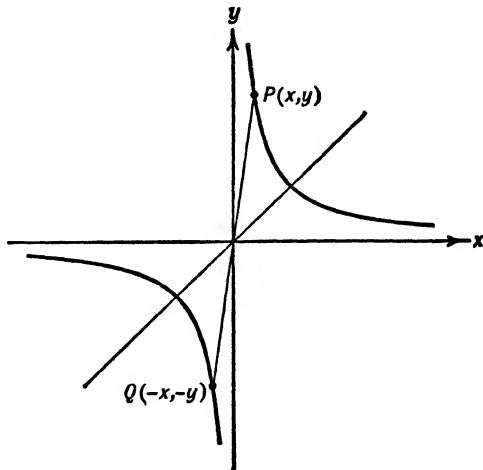


FIG. 98

116. The Rectangular Hyperbola. We next investigate the equation

$$xy = c$$

17-15

If the constant c is positive, x and y must have like signs, and the curve lies entirely in the first and third quadrants. If c is negative, x and y must have unlike signs, and the curve lies entirely in the second and fourth quadrants.

As x becomes small, y grows indefinitely large. As x becomes large, y approaches zero. But equation 17-15 cannot be satisfied if either x or y is zero, except in the degenerate case where $c = 0$. The coordinate axes are asymptotes of the curve. It will be shown later (Chapter 18) that equation 17-15 represents a hyperbola whose axis is inclined at 45° to the coordinate axes. Because the asymptotes are perpendicular to one another, it is called a rectangular hyperbola.

The points $P(x, y)$ and $Q(-x, -y)$ are said to be symmetrical with respect to the origin of axes (see Figure 98). If we replace x by $(-x)$ and y by $(-y)$, equation 17-15 remains unaltered. Hence to every point P on the rectangular hyperbola 17-15 there corresponds an image point Q also on the hyperbola. The curve is said to be symmetrical about the origin.

Evidently any curve that is symmetrical about both coordinate axes is symmetrical about the origin (for example, the ellipse 17-7 and the hyperbola 17-9). But the example just discussed shows that the converse is not true.

Exercises

1. Draw on one pair of axes the curves whose equations are

$$xy = 4 \quad xy = -4 \quad xy = 10 \quad xy = -10$$

2. Justify the statement that any curve that is symmetrical with respect to both coordinate axes is symmetrical about the origin.
3. Locate the foci for the hyperbolas of Exercise 1.

117. Variation. In this chapter, we have been studying the geometric properties of a certain class of curves. Our method has been to associate each curve with a particular equation, so that geometric problems could be formulated in the language of algebra, and solved by algebraic methods. We now digress briefly in order to consider, from another point of view, certain of the equations that have been encountered.

One of the commonest, and surely the simplest, way in which one quantity can depend upon another is illustrated by the law of Gay-Lussac: The volume of a gas varies directly as the absolute temperature. This law, or any other law of the same kind, is expressed symbolically by the equation

$$y = kx \qquad 17-16$$

where x is the temperature, and y the volume. An equivalent statement of the law of Gay-Lussac is the following: The volumes of equal amounts of a gas are directly proportional to the absolute temperatures. Symbolically, this is written

$$\frac{y_1}{y_2} = \frac{x_1}{x_2} \qquad 17-17$$

The forms 17-16 and 17-17 are completely equivalent; either can readily be obtained from the other. The first is simpler in algebraic form, the latter for arithmetic computation. *Direct variation* was discussed in Chapter 6.

Another simple way in which one quantity can depend upon another is called *inverse variation*. For example, Boyle's law is often stated thus: The volume of a gas varies inversely with the pressure. In symbols,

$$y = \frac{k}{x} \quad 17-18$$

where x now represents the pressure. This is equivalent to equation 17-15:

$$xy = c$$

It is evident that direct variation may be represented graphically by a straight line; while inverse variation is represented by a rectangular hyperbola.

Boyle's law may also be stated in alternative form as follows: The volume of a gas is inversely proportional to the pressure. In symbols,

$$\frac{y_1}{y_2} = \frac{x_2}{x_1} \quad 17-19$$

Another simple form of functional dependence is that in which one quantity varies as the square of another. For example, the power loss in an electrical circuit varies as the square of the current. This relationship is expressed thus:

$$y = kx^2 \quad 17-20$$

where x is the current, and y the power loss. If power loss is plotted against current, the resulting curve is a parabola. When equation 17-20 is expressed in proportion form, it becomes

$$\frac{y_1}{y_2} = \frac{x_1^2}{x_2^2} \quad 17-21$$

It is possible to write a formula which includes all of the foregoing forms of variation, namely

$$y = kx^n \quad 17-22$$

If $n = 1$, we have direct variation. If $n = -1$, the formula represents inverse variation; and so on. Any relationship of the form 17-22 is called a *power function*, because y varies as some power of x . The constant k ,

called the *constant of proportionality*, usually has an important physical meaning in any particular case.

Exercises

1. Derive equation 17-19, by assuming that (x_1, y_1) and (x_2, y_2) are points known to lie on the hyperbola 17-18.
2. Derive equation 17-21, by assuming that (x_1, y_1) and (x_2, y_2) are points known to lie on the parabola 17-20.
3. What curve results from equation 17-22 if $n = \frac{1}{2}$? State the law of variation in words for this case. Write the law of variation in proportion form.
4. The voltage drop between two points in an electrical circuit is directly proportional to the current flowing (Ohm's law). What is the meaning of the constant of proportionality in this case?

CHAPTER 18

TRANSFORMATION OF COORDINATES

118. Introduction. Linear transformations were employed in a systematic way by Vieta (1540–1603), perhaps the earliest algebraist whose creative work seems to modern mathematicians to have been of the first quality. Since Vieta's time, transformation theory has steadily increased in importance, until today it permeates almost all of mathematics.

In this chapter we shall discuss a certain class of linear transformations that occurs in the study of the translation and rotation of geometric objects. The somewhat abstract ideas of the transformation theory will be

applied to the conic sections, which will be shown to include all curves whose equations are of the second degree.

119. Translation of Axes. We may choose any desired position for the coordinate axes to which a given curve is referred. The equation which represents the curve is, of course, different for different positions of the coordinate axes. One reason for changing the position of the axes is to bring about a simplification of the equation of a given curve.

When the new axes are drawn parallel to the old, the change in position is called a translation of axes. Consider a point P whose coordinates are (x, y) with respect to one pair of coordinate axes, and (X, Y) with respect to a second pair, parallel to the first (Figure 99). The old coordinates are expressed in terms of the new by the equations

$$\begin{aligned}x &= X - h \\y &= Y - k\end{aligned}\tag{18-1}$$

The effect of replacing x by $X - h$ is to change the abscissa of every point in the plane; the new abscissa is measured from a new Y -axis which

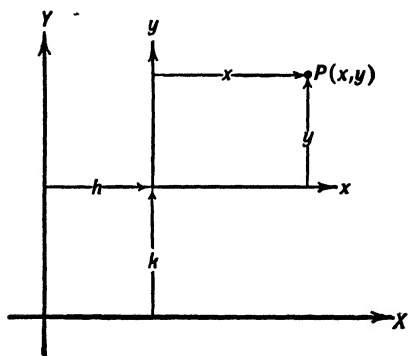


FIG. 99

is h units to the left (or to the right, if h is negative) of the old y -axis. A similar statement may be made as to the ordinate of every point in the plane. Hence the substitutions 18-1 transform the equation of any given curve into a new equation, which represents the same curve, referred to the new coordinate axes.

Example. A circle with center at the origin of the xy pair of axes is represented by equation 17-1:

$$x^2 + y^2 = r^2$$

Applying the equations of translation 18-1, the same circle is found to have the equation

$$(X - h)^2 + (Y - k)^2 = r^2$$

with respect to a new pair of axes. The coordinates of the center of the circle are (h,k) on the new system. The same equation (17-2) was found directly in the last chapter.

Exercises

It is suggested that new axes be drawn in color, and new coordinates written in the same color, in all diagrams. An alternative plan is to use pencil for one system, and ink for the other.

- Find the rectangular coordinates of the points $(4,4)$ and $(3,2)$, referred to new axes, parallel to the original axes, and intersecting at the point $(-6, -5)$ on the original system.
- Find the new coordinates of the points $(-3,5)$ and $(2,-6)$, when the axes are translated 2 units to the right, and 4 units down.
- Find the equation of the line $2x - 3y = 7$ when the coordinate axes are shifted 3 units to the left, and 4 units upward. *Ans.* $2X - 3Y = 25$
- Find the equation of the line $3x + 4y = 8$, referred to parallel axes through the point $(-2,5)$.
- Find the equations of the lines

$$x - 2y = 3$$

$$2x + 4y = 5$$

referred to parallel axes, chosen in such a way that the new equations possess no constant terms.

- By a translation of axes, remove the first degree terms from the equation

$$x^2 + y^2 + 6x - 4y = 3$$

120. Parabola with Vertex at (h,k) . The equation of a parabola with vertex at the origin of the xy system of coordinates is $y^2 = ax$, if the axis is horizontal; and $x^2 = ay$, if the axis is vertical. Let us find the corresponding equations referred to a new system (XY) , with respect to which the vertex is at the point (h,k) , as illustrated in Figure 100.

Application of the equations of translation 18-1 leads immediately to the standard forms

$$\begin{aligned}(Y - k)^2 &= a(X - h) && \text{axis horizontal} \\(X - h)^2 &= a(Y - k) && \text{axis vertical}\end{aligned}\tag{18-2}$$

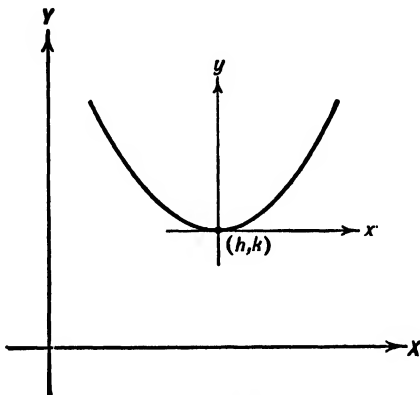


FIG. 100

If these equations be expanded, they assume the forms

$$\begin{aligned}Y^2 + DX + EY + F &= 0 \\X^2 + DX + EY + F &= 0\end{aligned}\tag{18-3}$$

121. Reduction to Standard Form. If we are confronted with an equation in one of the forms 18-3 (or in the corresponding form 17-3 for a circle), the geometrical constants of the curve are conveniently obtained by the method of completing the square (Chapter 12).

Example 1. The equation of a circle is

$$3x^2 + 3y^2 + 2x - 4y - 20 = 0$$

Let the equation be rewritten in the form

$$x^2 + \frac{2}{3}x + y^2 - \frac{4}{3}y = \frac{20}{3}$$

where blank spaces have been left, for convenience in completing the square on x and on y . Adding to both members the square of one-half the coefficient of x , and the square of one-half the coefficient of y , we obtain

$$x^2 + \frac{2}{3}x + \frac{1}{9} + y^2 - \frac{4}{3}y + \frac{4}{9} = \frac{20}{3} + \frac{1}{9} + \frac{4}{9}$$

which may be written

$$(x + \frac{1}{3})^2 + (y - \frac{2}{3})^2 = \frac{65}{9}$$

Hence $h = -\frac{1}{3}$, $k = \frac{2}{3}$, and $r = \sqrt{65}/3$. The center of the circle is the point $(-\frac{1}{3}, \frac{2}{3})$. The radius is approximately 2.7 units.

Example 2. The equation

$$2x^2 + 4x = 3y$$

may be written in the form

$$x^2 + 2x = \frac{3}{2}y$$

Upon completing the square on x , we find

$$x^2 + 2x + 1 = \frac{3}{2}y + 1$$

which is equivalent to

$$(x + 1)^2 = \frac{3}{2}(y + \frac{2}{3})$$

Hence $h = -1$, $k = -\frac{2}{3}$, and $a = \frac{2}{3}$. The equation is that of a parabola with vertical axis. The vertex is at the point $(-1, -\frac{2}{3})$. The focus is $\frac{1}{4}a = \frac{1}{6}$ units

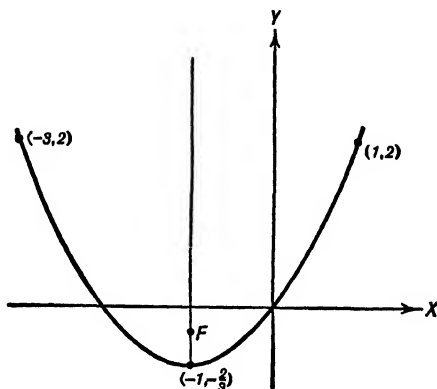


FIG. 101

above the vertex. In order to sketch the curve, it is convenient to locate another pair of points on it. Taking $x = 1$, we find $y = 2$. Hence the point $(1, 2)$ and the symmetrically situated point $(-3, 2)$ are on the parabola (Figure 101).

Exercises

Reduce the following six equations to standard forms, and draw the curves.

1. $x^2 + y^2 - 2x + 2y = 2$
2. $2x^2 + 2y^2 - 3x - 4y = 1$
3. $12y^2 - 24x - 36y + 11 = 0$
4. $6x^2 + 8 = 3(y + 4x)$
5. $3x^2 + 4x - 6y = 3$
6. $2y^2 + 5x + 5y - 1 = 0$
7. Find the equation of the parabola

$$y^2 + 2x + 10 = 4y$$

referred to parallel axes through the vertex.

8. Find the equation of the parabola

$$2x^2 + 4x + 8 = 3y$$

referred to parallel axes through the vertex.

9. Find graphically the points of intersection of the curves

$$y = x + 3$$

$$y^2 + 1 = x + 4y$$

and check by solving the equations simultaneously.

10. As in Exercise 9 for the curves

$$2x + y = 4$$

$$2x^2 = 4x + y$$

11. As in Exercise 9 for the curves

$$4x + 3y = 25$$

$$x^2 + y^2 = 8x + 6y$$

12. As in Exercise 9 for the curves

$$2x + y = 5$$

$$3x^2 + 3y^2 = 2x + y + 1$$

122. Ellipse with Center at (h,k) . The equation of an ellipse with center at the origin of the xy system of coordinates is (equation 17-7)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let us find the equation of the same curve, referred to a new (XY) set of axes. Let the center of the ellipse be the point (h,k) on the new system. Setting

$$x = X - h$$

$$y = Y - k$$

the equation, referred to the new axes, becomes

$$\frac{(X - h)^2}{a^2} + \frac{(Y - k)^2}{b^2} = 1 \quad 18-4$$

The expanded form of this equation is

$$AX^2 + CY^2 + DX + EY + F = 0 \quad 18-5$$

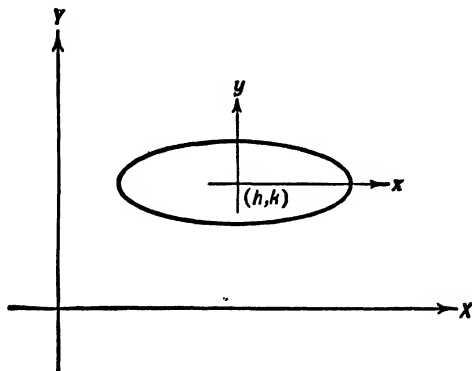


FIG. 102

The coefficients A and C must have like signs.

Example. Let us reduce to standard form the equation

$$8x^2 + 3y^2 - 16x + 2y = 0$$

The terms are rearranged thus:

$$8(x^2 - 2x) + 3(y^2 + \frac{2}{3}y) = 0$$

Completing the square upon x and y , we obtain

$$8(x^2 - 2x + 1) + 3(y^2 + \frac{2}{3}y + \frac{1}{9}) = 8 + \frac{1}{3}$$

That is,

$$8(x - 1)^2 + 3(y + \frac{1}{3})^2 = \frac{25}{3}$$

Dividing both members by $\frac{25}{3}$, we arrive at

$$\frac{(x - 1)^2}{\frac{25}{24}} + \frac{(y + \frac{1}{3})^2}{\frac{25}{9}} = 1$$

Thus $a = 5/2\sqrt{6}$, $b = \frac{5}{3}$, $h = 1$, and $k = -\frac{1}{3}$. The given equation represents an ellipse with center at $(1, -\frac{1}{3})$, horizontal semi-diameter approximately 1.02 units, and vertical semi-diameter $\frac{5}{3}$ units in length.

123. Hyperbola with Center at (h,k) . The equations (17-9 and 17-14) representing a hyperbola with center at the origin are $x^2/a^2 - y^2/b^2 = 1$, if the axis of the hyperbola is horizontal; and $y^2/b^2 - x^2/a^2 = 1$, if the axis is vertical. Referred to the XY set of axes, these equations become

$$\frac{(X-h)^2}{a^2} - \frac{(Y-k)^2}{b^2} = 1 \quad \text{axis horizontal} \quad 18-6$$

$$\frac{(Y-k)^2}{b^2} - \frac{(X-h)^2}{a^2} = 1 \quad \text{axis vertical}$$

The expanded form of both of these equations is

$$AX^2 - CY^2 + DX + EY + F = 0 \quad 18-7$$

This equation differs from that of an ellipse (equation 18-5) in that the coefficients of the second degree terms are unlike in sign.

Example. Reduce to standard form the equation

$$x^2 - 4y^2 + 6x + 8y + 9 = 0$$

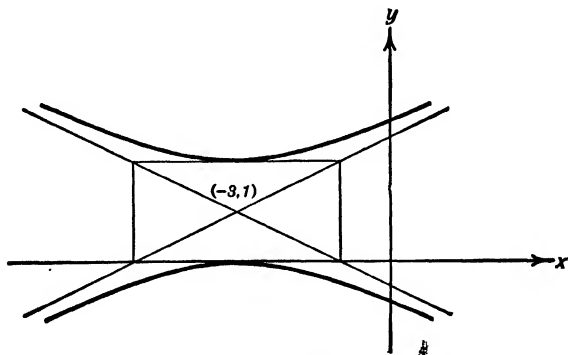


FIG. 103

Rearranging terms and completing the square, we have

$$(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = -9 + 9 - 4$$

Dividing both members by -4 , the result may be written

$$\frac{(y-1)^2}{1} - \frac{(x+3)^2}{4} = 1$$

The equation represents a hyperbola with vertical axis, and center at $(-3,1)$. See Figure 103.

Exercises

Reduce the following eight equations to standard forms, and draw the curves.

1. $4x^2 + 3y^2 - 16x + 6y = 17$
2. $16x^2 + 25y^2 + 64x + 150y + 285 = 0$
3. $x^2 + 4y^2 + 5x - 3y = 0$
4. $3x^2 + y^2 + 4x + 6y = 7$
5. $16x^2 - 25y^2 - 32x + 100y = 484$
6. $16x^2 - 25y^2 - 32x + 100y + 316 = 0$
7. $4x^2 - 8y^2 + 12x - 16y + 9 = 0$
8. $x^2 - 2y^2 + 3x - 4y = 0$
9. Show that the equation

$$(X - h)(Y - k) = \text{constant}$$

represents a rectangular hyperbola with center at (h,k) .

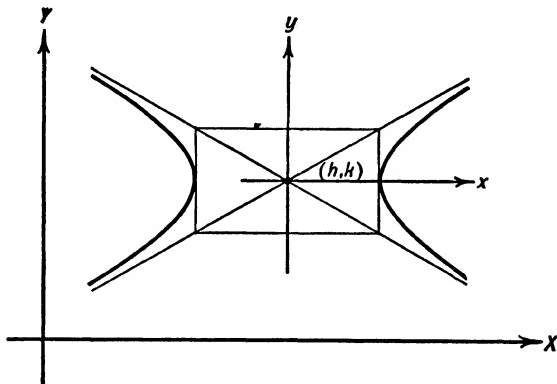


FIG. 104

10. Reduce to the form given in Exercise 9, and draw, the rectangular hyperbola

$$xy + 3x = 2y + 15$$

11. Reduce to standard form, and draw, the rectangular hyperbola

$$xy + ay = 2ax$$

124. The Equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$. In applying our transformation theory to the conics, we have paved the way towards a general theory which includes the various conics as special cases. Consider the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \quad 18-8$$

where the coefficients A, C, \dots , are in the field of real numbers. We may distinguish the following principal cases:

Case I. If either A or C is zero, the equation represents a parabola.

Case II. If A and C are both positive (or both negative), the equation represents an ellipse. If, in particular, $A = C$, the equation represents a circle.

Case III. If A and C have unlike signs, the equation represents a hyperbola.

The presence of the first degree terms (Dx and Ey) indicates that the center of the figure does not lie at the origin of axes. In Cases II and III above, the first degree terms may be removed by the transformations

$$x = X - \frac{D}{2A}$$

$$y = Y - \frac{E}{2C}$$

In the parabolic case (where one of the terms of the second degree is missing) it is not possible to remove both terms of the first degree by a translation of axes. Assuming that $A = 0$, we may reduce the equation to standard form as follows:

$$C \left(y^2 + \frac{E}{C} y \right) = -Dx - F$$

$$C \left(y^2 + \frac{E}{C} y + \frac{E^2}{4C^2} \right) = -Dx - F + \frac{E^2}{4C}$$

$$C \left(y + \frac{E}{2C} \right)^2 = -D \left(x + \frac{F}{D} - \frac{E^2}{4CD} \right)$$

Hence the transformation

$$x = X + \frac{E^2}{4CD} - \frac{F}{D}$$

$$y = Y - \frac{E}{2C}$$

reduces equation 18-8 (provided that $A = 0$) to the form

$$CY^2 + DX = 0$$

The constant term, as well as the first degree term in y , have been removed. A similar treatment is readily devised when the second degree term in y is missing.

Thus we see that any function of the form 18-8 can be reduced by a translation of axes to one or other of the five forms

$$\text{Case I: } X^2 = aY \text{ or } Y^2 = aX$$

$$\text{Case II: } \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

$$\text{Case III: } \frac{X^2}{a^2} - \frac{Y^2}{b^2} = \pm 1$$

Exercises

Identify the curve represented by each of the following six equations, find a transformation which reduces each to simplest form, and draw the curve, showing both the old and the new axes.

1. $x^2 + 13 = 2x + 4y$
2. $5y^2 + 20y + 22 = x$
3. $3x^2 + 4y = y^2 + 6x + 109$
4. $x^2 + 4y^2 + x + 9 = 12y$
5. $4x^2 + y^2 - 71 = 8x + 10y$
6. $9y^2 + 20x = 4x^2 + 6y + 25$
7. Find a transformation which will reduce the equation $By^2 + Cx + Dy + E = 0$ to one of the five forms listed in the text.

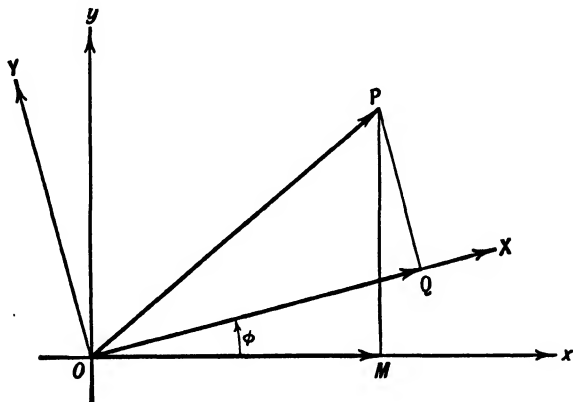


FIG. 105

125. Rotation of Axes. We next seek to determine the transformation equations corresponding to a change of coordinate axes in which the origin remains fixed, and the new axes are inclined to the old at an angle ϕ (Figure 105). Let P be any point in the plane. Its coordinates (x, y) are the components of the line vector OP on the old pair of axes. The new co-

ordinates (X, Y) are the components of the same line vector on the new pair of axes. The situation is similar to that discussed in Chapter 10 (see Figure 36, page 105). The component of OP along OX is equal to the sum of the projections of OM and MP along OX . That is,

$$OQ = OM \cos \phi + MP \sin \phi$$

from which we see that

$$X = x \cos \phi + y \sin \phi \quad 18-9$$

In like manner, we readily find

$$Y = -x \sin \phi + y \cos \phi \quad 18-10$$

These equations express the new coordinates in terms of the old.

The same reasoning might be employed in finding the equations which express the old coordinates in terms of the new. If, however, we notice that the desired equations are equivalent to 18-9 and 18-10, provided that the angle of rotation be taken as $-\phi$, we may write immediately

$$x = X \cos (-\phi) + Y \sin (-\phi)$$

$$y = -X \sin (-\phi) + Y \cos (-\phi)$$

from which we have

$$x = X \cos \phi - Y \sin \phi \quad 18-11$$

$$y = X \sin \phi + Y \cos \phi \quad 18-12$$

Equations 18-11 and 18-12 represent the *direct* transformation from the old (xy) axes to the new (XY) axes. Equations 18-9 and 18-10 represent the *inverse* transformation from the new system back to the old system.

Example 1. Let the equation

$$x^2 = ay$$

be referred to new axes, which are rotated through 90° from the position of the old system. Taking $\phi = 90^\circ$, whence $\cos \phi = 0$ and $\sin \phi = 1$, the equations of transformation become

$$x = -Y$$

$$y = X$$

Hence the new equation is (see Figure 106)

$$Y^2 = aX$$

Similarly, the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

when referred to new axes, advanced 90° from the old pair, becomes

$$\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$$

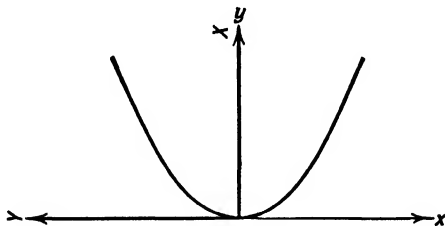


FIG. 106

Hence we see that by a translation of axes, followed (if necessary) by a rotation through 90° , any function of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \quad 18-8$$

can be reduced to one or other of three canonical forms:

$$X^2 = aY \quad \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \quad \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \quad 18-13$$

Example 2. Find the equation of the curve

$$xy = c$$

referred to new axes, inclined at an angle of 45° to the old system.

Taking $\phi = 45^\circ$, the equations of transformation become

$$x = \frac{1}{\sqrt{2}} (X - Y)$$

$$y = \frac{1}{\sqrt{2}} (X + Y)$$

Hence the equation of the curve, referred to the new axes, is

$$\frac{1}{\sqrt{2}} (X - Y) \left(\frac{1}{\sqrt{2}} (X + Y) \right) = c$$

which reduces to

$$X^2 - Y^2 = 2c$$

The curve is a hyperbola, whose asymptotes are the old (xy) axes (Figure 107).

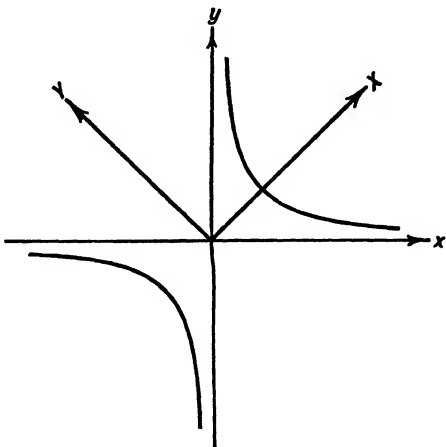


FIG. 107

126. The General Equation of the Second Degree. If we make the convention that the degree of a term is the sum of the powers of the variables occurring in it, the general form for an equation of the second degree (in two variables) is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad 18-14$$

We shall show that, by a rotation of axes, the cross-product term Bxy can be removed, and the equation reduced to the form 18-8. From this result it follows that any equation of the second degree (in two variables) represents an ellipse, a hyperbola, or a parabola. The occurrence of a cross-product term thus indicates that the axis of the conic is oblique to the coordinate axis system. By a suitable choice of coordinate axes, any equation of the form 18-14 can be obtained in one or other of the canonical forms 18-13.

To show this, we first notice that, upon rotating the axes through an angle ϕ , the new equation obtained from 18-14 has for its cross-product term

$$(2A \sin \phi \cos \phi - 2C \sin \phi \cos \phi - B \cos^2 \phi + B \sin^2 \phi)XY$$

The remaining terms in the new equation are of no interest for the moment. We wish to ascertain whether there exists a value of ϕ for which the new cross-product term vanishes. We therefore endeavor to solve the equation

$$2(A - C) \sin \phi \cos \phi - B(\cos^2 \phi - \sin^2 \phi) = 0$$

Using the trigonometric identities

$$\sin 2\phi = 2 \sin \phi \cos \phi$$

$$\cos 2\phi = \cos^2 \phi - \sin^2 \phi$$

we have immediately

$$(A - C) \sin 2\phi = B \cos 2\phi \quad 18-15$$

$$\therefore \tan 2\phi = \frac{B}{A - C} \quad 18-16$$

Equation 18-16 provides the desired transformation, which removes the cross-product term from equation 18-14, for every set of real values of A , B , and C , unless $A = C$. This case is discussed in the following example.

Example. Consider the curve whose equation is

$$x^{1/2} + y^{1/2} = a^{1/2}$$

The equation, in its present form, is not of the second degree. However, upon squaring twice, the equation becomes

$$x^2 - 2xy + y^2 - 2ax - 2ay + a^2 = 0$$

In this example, $A = C = 1$. Equation 18-16 fails, because the step by which it was obtained involves division by $A - C$, which is not a permissible algebraic operation when $A - C = 0$. However, from equation 18-15 we have

$$0 = -2 \cos^2 \phi$$

$$\therefore 2\phi = 90^\circ$$

$$\phi = 45^\circ$$

The equations of transformation become

$$x = \frac{1}{\sqrt{2}}(X - Y)$$

$$y = \frac{1}{\sqrt{2}}(X + Y)$$

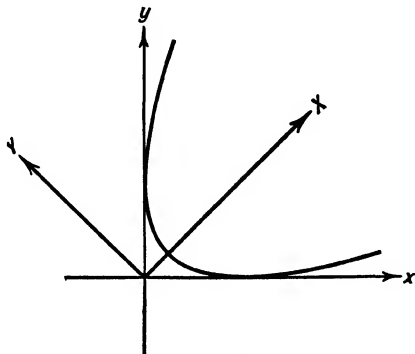


FIG. 108

The new equation of the curve is

$$Y^2 = \sqrt{2}a \left(X - \frac{a}{2\sqrt{2}} \right)$$

The curve is drawn in Figure 108.

Exercises

By a rotation of axes, remove the xy term from the following four equations, and draw the curves, showing the original and also the new systems of coordinate axes.

1. $2x^2 + 2xy + 2y^2 - 1 = 0$
2. $2xy - x^2 - y^2 + 2x + 2y = 1$
3. $2x + 3y + \sqrt{2}xy + \sqrt{2} = 0$
4. $3x^2 + 2xy + 3y^2 + 8x = 8y + 4$

CHAPTER 19

POLAR COORDINATES. PLANE CURVES

127. Polar Coordinates. It is possible to specify any point P in a plane by stating: (a) the distance r from a fixed point O ; and (b) the angle θ made by OP with some fixed direction (Figure 109). It is convenient to lay off the angle θ in the standard way (see Chapter 4). The quantities (r, θ) are called *polar coordinates* of the point P . The fixed point O is called the pole.

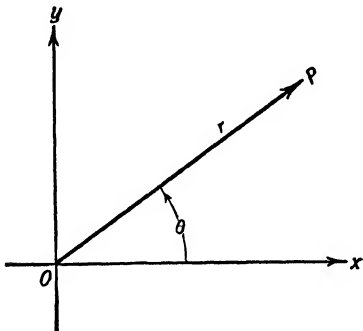


FIG. 109

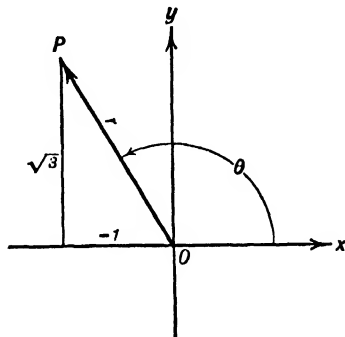


FIG. 110

Example. Let us find the polar coordinates of a point whose cartesian coordinates are $x = -1$, $y = \sqrt{3}$ (Figure 110).

From the figure, $r = 2$, and $\tan \theta = -\sqrt{3}$. Hence the point P has the polar coordinates $(2, 120^\circ)$. Of course, there are infinitely many angles corresponding to any position of the radius vector OP ; so that any of the pairs $(2, -240^\circ)$, $(2, 480^\circ)$, $(2, 840^\circ)$, may be taken to be the polar coordinates of P . Moreover, if negative values of r be interpreted as meaning *distance measured backward along the radius vector produced*, then $(-2, -60^\circ)$ is another pair of polar coordinates of P .

Exercises

1. Plot the points whose polar coordinates are $(3, 70^\circ)$, $(5, 215^\circ)$, $(4, -45^\circ)$, $(-6, 70^\circ)$.
2. Plot the points whose polar coordinates are $(\sqrt{2}, \pi/4)$, $(5, \pi)$, $(-4, -\pi/3)$. (The angles are expressed in radians.)
3. Write the cartesian coordinates of the points of Exercise 2 above.

4. Find three pairs of polar coordinates for the point whose cartesian coordinates are $(-2, -2)$.
5. Where do all the points lie for which $r = 10$?
6. Where do all the points lie for which $\theta = \pi/6$?
7. If $(3, \pi/6)$ are polar coordinates of a point P , what are the polar coordinates of a point symmetrically situated with respect to a vertical line through the pole O ? A horizontal line through O ? What are the coordinates of a point symmetrically situated with respect to O ?

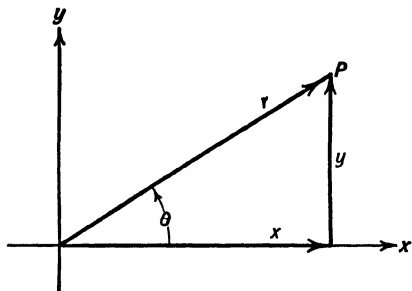


FIG. 111

128. Changing from Polar to Cartesian Coordinates. In both coordinate systems, any point P may be considered to determine a certain vector OP , and, conversely, the vector may be considered to specify a certain point. When we prefer to work with the *magnitude* r , and the *angle* θ of the vector, we are employing polar coordinates of the point. When we work with the

components of the vector, we are using the cartesian coordinates of the point.

In Chapter 1 of this book there was discussed the relation between a vector and its components. For our present purpose, we may write (see Figure 111)

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \tag{19-1}$$

These equations make it easy to change from cartesian to polar coordinates. Again, from Figure 111 we may write

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tan \theta &= \frac{y}{x} \end{aligned} \tag{19-2}$$

Equations 19-2 are used in changing from polar to cartesian coordinates.

Example 1. Find the polar equation of the circle $x^2 + y^2 = 2ax$.

From equations 19-1, we have

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 2ar \cos \theta$$

Factoring the left-hand member, we obtain

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 2ar \cos \theta$$

In simplest form, this becomes

$$r = 2a \cos \theta$$

(In removing the common factor r , the curve whose equation is $r = 0$ has been discarded. How may this step be justified?)

The student should observe that the same circle (Figure 112) is represented in polar coordinates by a trigonometric function, and in cartesian coordinates by an algebraic equation of the second degree. These two functions, both representing the same curve, are quite different in appearance.

Example 2. It is required to draw the curve whose equation is $r = 4 \csc \theta$.

On the chance that the cartesian equation of the curve will be a familiar type, let us change to cartesian coordinates. From Figure 111, we see that

$$\begin{aligned} \csc \theta &= \frac{1}{\sin \theta} \\ &= \frac{r}{y} \\ &= \frac{\sqrt{x^2 + y^2}}{y} \end{aligned}$$

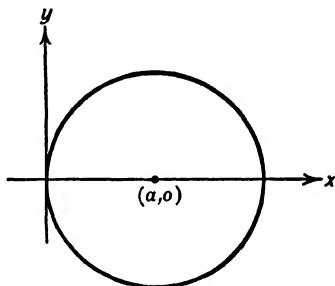


FIG. 112

Hence the cartesian equation of the curve is

$$\sqrt{x^2 + y^2} = \frac{4\sqrt{x^2 + y^2}}{y}$$

which readily simplifies to

$$y = 4$$

We recognize the curve as a straight line, parallel to the x -axis, and lying 4 units above it.

Exercises

- Find the cartesian equation of the curve

$$r(1 - \cos \theta) = 3$$

Draw the curve.

- Find the cartesian equation of the curve

$$r(2 - \cos \theta) = 2$$

Draw the curve.

- Find the polar equation of the straight line

$$y = 3x - 2$$

- Find the polar equation of the parabola

$$4y^2 = x$$

Draw the curve.

5. Find the cartesian equation of a circle with center in the second quadrant, and radius a , tangent to both coordinate axes. Find the equation of the same circle in polar coordinates.
6. Find the cartesian equation of a circle with center in the third quadrant, and radius a , tangent to both coordinate axes. Find the equation of the same circle in polar coordinates.

129. The Conics. For many of the problems of applied mathematics, the conics are more conveniently expressed in polar than in rectangular coordinates.

Let us obtain the equation of a curve described by a point which moves in such a way that its distance from a fixed point is directly proportional to its distance from a given line. It is best to choose the fixed point as the pole. The given line is called the directrix; it may without loss of generality be taken to be a vertical line passing through the point (k, π) , as in Figure 113.

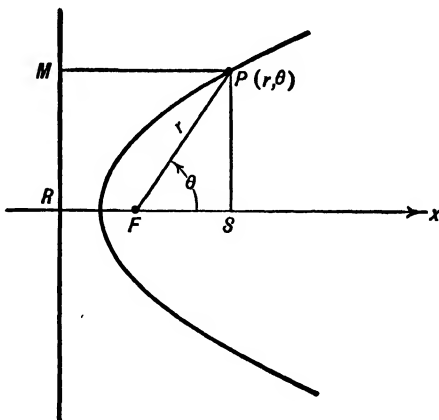


FIG. 113.

By hypothesis,

$$FP = e(MP)$$

where e is the constant of proportionality. But

$$\begin{aligned} MP &= RS \\ &= RF + FS \\ &= k + r \cos \theta \end{aligned}$$

Hence the equation of the curve is

$$r = e(k + r \cos \theta)$$

which is linear in r . Upon solving for r , the result is found to be

$$r = \frac{ek}{1 - e \cos \theta} \quad 19-3$$

Upon changing to rectangular coordinates, we find that the equation takes the form

$$x^2 + y^2 = e^2(k + x)^2$$

Hence the curve is a conic. The constant e is called the *eccentricity*. If $e = 1$, the curve is a parabola. If e is greater than 1, the curve is a hyperbola. If e is less than 1, the curve is an ellipse.

Exercises

1. Draw the curve

$$r = \frac{2}{1 - \cos \theta}$$

after changing to rectangular coordinates.

2. Draw the curve

$$r = \frac{2}{2 - \cos \theta}$$

after changing to rectangular coordinates.

3. Prove that if $e = 1$, equation 19-3 represents a parabola.
4. Prove that if e is less than 1, equation 19-3 represents an ellipse.
5. Show that

$$c = ae$$

where $c^2 = a^2 - b^2$, if e is less than 1; and $c^2 = a^2 + b^2$, if e is greater than 1.

6. Show that the pole is a focal point of the conic 19-3.

130. Plotting Curves in Polar Coordinates. It may happen that the cartesian equation of a curve is too complicated to be useful, so that it is best to work directly from the equation in polar coordinates. The simplest plan is to construct a table of values of r and θ , and plot points until the shape of the curve becomes evident.

Example. It is required to draw the curve

$$r = 4 \cos 2\theta$$

If we set $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ (equation 10-13), it is not difficult to

obtain the cartesian equation of the curve; it is

$$\sqrt{x^2 + y^2} = 4 \left(\frac{x^2 - y^2}{x^2 + y^2} \right)$$

But the cartesian form is not promising; it is better to employ polar coordinates.

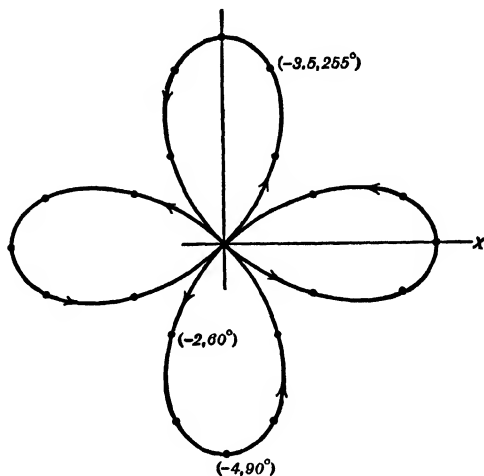


FIG. 114

If θ be taken at intervals of 45° , it will be found that the shape of the curve is not clearly revealed. It is better to take an interval of 15° ; thus

$\theta = 0^\circ$	15°	30°	45°	60°	75°	90°	105°	120°
$r = 4$	3.5	2	0	-2	-3.5	-4	-3.5	-2

and so on. The curve, a four-petaled rose, is shown in Figure 114, with points computed from $\theta = 0^\circ$ to $\theta = 360^\circ$. For larger values (and negative values) of θ , the same curve is re-traced.

Exercises

1. Draw the curve whose equation is

$$r = 6 \sin 2\theta$$

2. Show that if, when θ is replaced by $(-\theta)$, the polar equation of a curve is unaltered, the curve is symmetrical about the x -axis.
3. Draw the cardioid

$$r = 4(1 + \cos \theta)$$

4. Draw the limaçon

$$r = 2(2 - \sin \theta)$$

5. Draw the spiral

$$\pi r = \theta$$

(The angle should be expressed in radians.)

6. Draw the hyperbolic spiral

$$r\theta = \pi$$

131. The Locus of an Equation. We have studied the geometrical properties of many curves by the device of finding equations to represent them. We now turn our attention to the device itself: it is perhaps the most important idea in analytic geometry.

It is understood that the word *equation* refers to any functional relationship between two quantities which can be expressed in the form of an equation. If there are any real pairs of values of the two quantities which satisfy the equation, then the geometrical configuration obtained by plotting all such pairs of values is called the *locus of the equation*.

The same idea may be expressed in another way. An equation is said to represent a given locus if, and only if, the following two conditions are met:

- (a) The coordinates of every point on the locus satisfy the equation.
- (b) Corresponding to every pair of values of x and y that satisfy the equation, there is a point which lies on the locus.

Example 1. Find the locus of the equation

$$(x^2 + y^2 + 1)(x^2 + y + 1) = 0$$

Any pair of values satisfying the equation must reduce at least one factor to zero, and, conversely, any pair of values reducing either factor to zero must satisfy the equation. Hence the locus may be separated into two parts, represented by the equations

$$x^2 + y^2 = -1$$

$$x^2 = -(y + 1)$$

The first equation has no real locus. The second represents a parabola with vertical axis, and vertex at $(0, -1)$, opening downward.

Example 2. Find the locus of the equation

$$x^2 - xy - 2y^2 + x + 4y - 2 = 0$$

The locus is a conic with a tilted axis. If the equation can be written as the product of two linear rational factors, the locus is easily obtained. If it has no rational factors, we must rotate the axes to eliminate the xy term, and proceed

as in Chapter 18. The method of testing for rational factors was described in Chapter 12; a necessary and sufficient condition is that the discriminant be a perfect square. Taking the equation as a quadratic in x , we may write it in the form

$$x^2 + (1 - y)x + (4y - 2y^2 - 2) = 0$$

Hence the discriminant

$$\begin{aligned} b^2 - 4ac &= 1 - 2y + y^2 - 16y + 8y^2 + 8 \\ &= (3y - 3)^2 \end{aligned}$$

The roots of the equation are

$$\begin{aligned} x &= \frac{-(1 - y) + (3y - 3)}{2} = 2y - 2 \\ x &= \frac{-(1 - y) - (3y - 3)}{2} = -y + 1 \end{aligned}$$

The equation may now be written in factored form:

$$(x - 2y + 2)(x + y - 1) = 0$$

Hence the locus of the equation is the pair of straight lines whose equations are

$$\begin{aligned} x - 2y + 2 &= 0 \\ x + y - 1 &= 0 \end{aligned}$$

Example 3. Sketch the curve represented by the equation

$$y = x + \frac{1}{x}$$

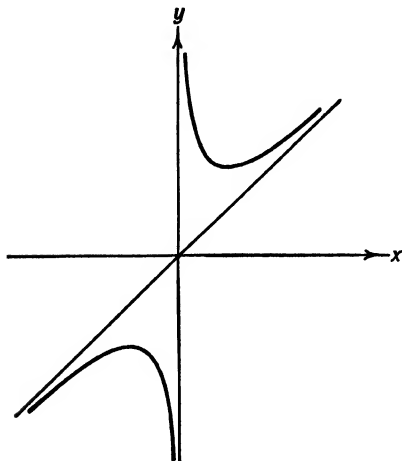


FIG. 115

Upon removing fractions, the equation is seen to be of the second degree. Hence the curve is a conic. Instead of reducing it to standard form by the transformations of Chapter 18, let us see what information may be obtained directly from the given equation.

When x is positive, so is y . When x is negative, y is negative also. Hence the curve lies entirely in the first and third quadrants.

When x is numerically small, the term $1/x$ becomes very large. There is no finite point on the curve for which $x = 0$; hence the y -axis is an asymptote of the curve.

When x is numerically large, the term $1/x$ becomes relatively small, and the curve resembles the straight line $y = x$. This line is another asymptote. The curve is drawn in Figure 115.

Exercises

Find the locus of each of the following five equations.

1. $6x^2 - 7xy - 3y^2 - x + 7y - 2 = 0$

2. $4x^2 = (y - 1)^4$

3. $y = \frac{4x}{1 - x}$

4. $y = \frac{x^3}{3} - 2$

5. $y = \frac{1}{1 - x} - \frac{1}{1 + x}$

6. If both members of an equation are divided by a common factor containing x , or y , or both, what is the effect upon the locus of the equation?

132. Finding the Equation of a Locus. In order to write the equation that represents a curve, it is necessary first to find a geometric condition satisfied by every point on the curve, and then to express this geometric condition in algebraic form.

Example 1. The distance from the line $x = -4$ of every point on a curve is twice the distance from the point $(2, 0)$. Find its equation.

The geometric condition is

$$PM = 2PF$$

From Figure 116, this equation is readily expressed in terms of x and y .

$$x + 4 = 2\sqrt{(x - 2)^2 + y^2}$$

Squaring both members, and reducing to type form, we obtain

$$\frac{(x - 4)^2}{16} + \frac{y^2}{12} = 1$$

which is an ellipse.

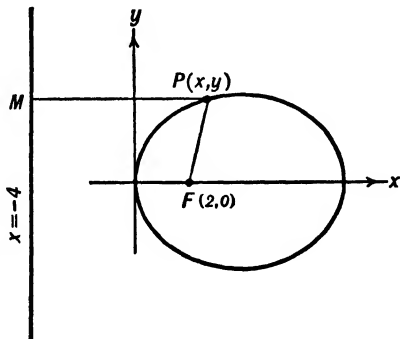


FIG. 116

Example 2. In Figure 117, point B moves along one edge of the box, from C to E , at a uniform rate of 2 units per second. At the same time, point D moves from E to F at the rate of 3 units per second. What is the equation of the locus of the point of intersection of AB and CD ?

Let us choose coordinate axes along AC and AF . Then

$$BC = 2t \text{ and } ED = 3t$$

where t is the number of seconds after starting. Using the properties of similar triangles, we have

$$\frac{x}{12} = \frac{y}{2t}$$

and

$$\frac{y}{8} = \frac{12 - x}{3t}$$

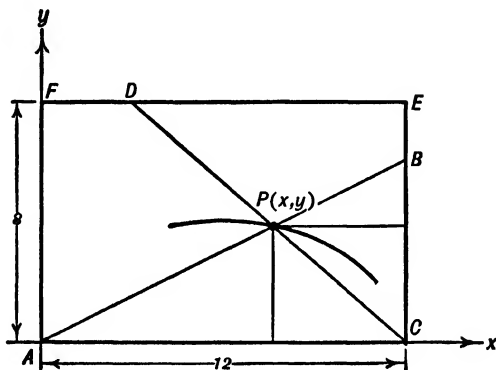


FIG. 117

In order to eliminate the unwanted variable t , let the first of these equations be divided by the second.

$$\frac{2x}{3y} = \frac{3y}{2(12 - x)}$$

Removing fractions, and reducing to standard form, the result is

$$\frac{(x - 6)^2}{36} + \frac{y^2}{16} = 1$$

Hence the locus of $P(x, y)$ is a quadrant of an ellipse, whose major diameter is AC .

Exercises

1. The point $P(x, y)$ moves so that it is always equidistant from the points $(-2, 4)$ and $(3, 3)$. Find the equation of the locus.
2. Find the equation of the locus such that all points on it are equidistant from the points $(1, -2)$ and $(-5, -8)$.

3. The distance from the point $(0,4)$ of every point on a curve is twice the distance from the line $y + 2 = 0$. Find the equation, and draw the curve.
4. The difference of the distances from any point on a curve to the points $(5,0)$ and $(-5,0)$ is constant, and equal to 6 units. Find the equation, and draw the curve.

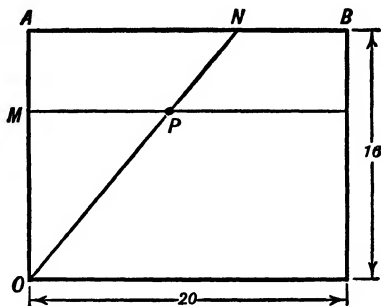


FIG. 118

5. In Figure 118, point M travels at a uniform rate of 4 units per second from O to A . At the same time, point N travels at a rate of 5 units per second from A to B . The point P is the intersection of a line ON with a horizontal line through M . Find the equation of the locus described by P , and name the curve.
6. A circle of radius 6 units is projected on a plane inclined at an angle of 60° to the plane of the circle. Find the equation of the curve thus obtained, and show that it is an ellipse.
7. Find the equation of the curve, such that the product of the distances of every point on it from the lines $x - y + 1 = 0$ and $x + y - 3 = 0$ is always 2. Reduce the equation to standard form, and draw the curve.
8. A loop of string, 25 inches in length, is passed around two nails which have been driven into a flat board, and is drawn taut by a pencil attached to one point of the string. The pencil and nails serve to locate three vertices of a triangle, of which the string represents the perimeter. Find the locus of the path described by the pencil, if the nails are 12 inches apart.
9. A point moves so that the square of its distance from $(4, -1)$ is always twice its distance from the y -axis. Find the equation of the locus, and draw the curve.

133. Sine Waves. In the study of wave phenomena, which are prominent in every major division of physical science, it is useful to possess some facility in sketching the graphs of sine and cosine functions.

Example 1. Sketch the sine wave

$$y = \sin x$$

Taking the angle at intervals of 30° , we construct the table

$x = 0^\circ$	30°	60°	90°	120°	150°	180°	210°
$y = 0$	0.50	0.87	1.00	0.87	0.50	0	-0.50

and so on. From 180° to 360° , the values of y repeat those from 0° to 180° , except for sign, since

$$\sin(180^\circ + x) = -\sin x$$

The graph of one cycle (from 0° to 360°) is shown in Figure 119.

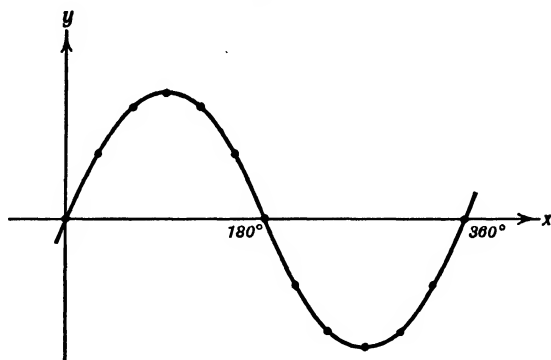


FIG. 119

From 360° to 720° , the values of y repeat those from 0° to 360° , since

$$\sin(360^\circ + x) = \sin x$$

For negative angles, the values of y repeat those for positive angles, except for sign, since

$$\sin(-x) = -\sin x$$

The sine wave is sketched, for an interval covering several cycles, in Figure 120. The wave extends indefinitely along the x -axis, in both directions.

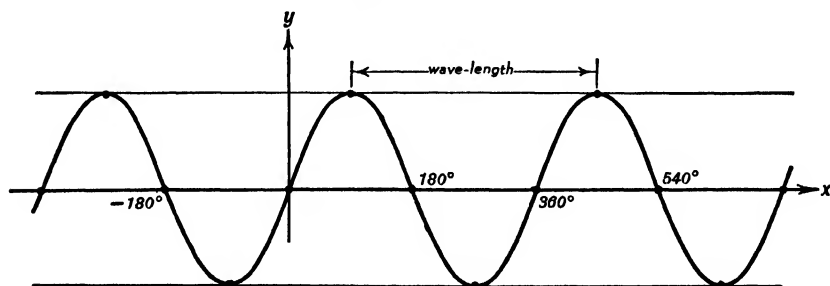


FIG. 120

Every undistorted sine or cosine wave has the form shown in Figure 120, except for scale factors. To sketch such a wave quickly, it is sufficient to locate the zero

points, and the peak and valley points, for one cycle. The curve is then easily extended to the right and left.

Example 2. Sketch the sine wave

$$y = A \sin x$$

taking the angle in radians.

The table of values for one cycle is

x	y
0	0
$\frac{\pi}{2}$	A
π	0
$\frac{3\pi}{2}$	$-A$
2π	0

The peak and valley points lie on lines located A units above and below the x -axis. The graph is shown in Figure 121. It may be obtained from Figure 120

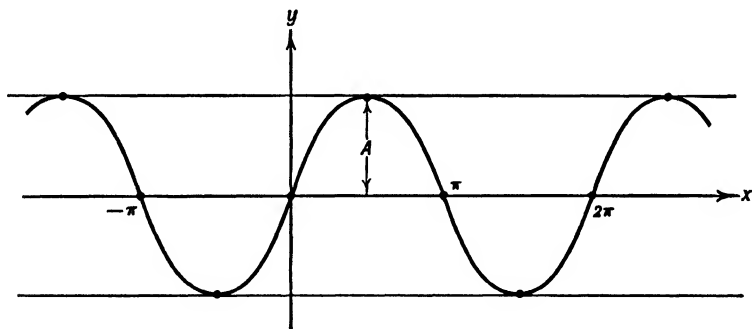


FIG. 121

by multiplying each ordinate by the scale factor A . The constant A is called the *amplitude* of the wave. In physical applications, it is an indication of wave strength, or intensity.

Example 3. Sketch the sine wave

$$y = \sin \omega x$$

taking the angle in radians.

The table of values for one cycle is

ωx	x	y
0	0	0
$\frac{\pi}{2}$	$\frac{\pi}{2\omega}$	1
π	$\frac{\pi}{\omega}$	0
$\frac{3\pi}{2}$	$\frac{3\pi}{2\omega}$	-1
2π	$\frac{2\pi}{\omega}$	0

The graph is shown in Figure 122. It is identical with Figure 120, except that distances along the x -axis have been divided by the scale factor ω . The constant

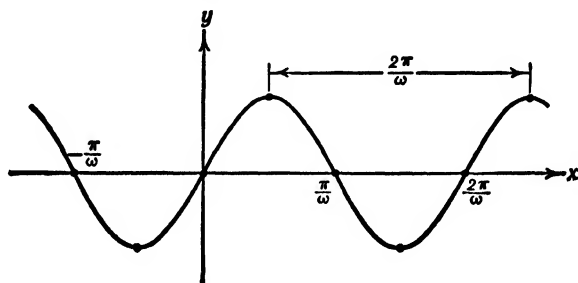


FIG. 122

ω is inversely proportional to the wave length, λ , which is the distance between crests.

$$\lambda = \frac{2\pi}{\omega}$$

19-4

In many physical applications, x represents the time in seconds, and the quantity $\omega/2\pi$ is called the *frequency*, or number of complete cycles per second. Musical sounds are produced by combinations of wave trains. Pitch is determined by the frequency of the waves, and loudness by their amplitude. If light is considered to be propagated by electromagnetic waves, color is determined by frequency, and brightness by amplitude.

Example 4. Sketch the sine wave

$$y = \sin(x - \phi)$$

taking the angle in radians.

The quarter-cycle points are given in the table

$x - \phi$	x	y
0	ϕ	0
$\frac{\pi}{2}$	$\frac{\pi}{2} + \phi$	1
π	$\pi + \phi$	0
$\frac{3\pi}{2}$	$\frac{3\pi}{2} + \phi$	-1
2π	$2\pi + \phi$	0

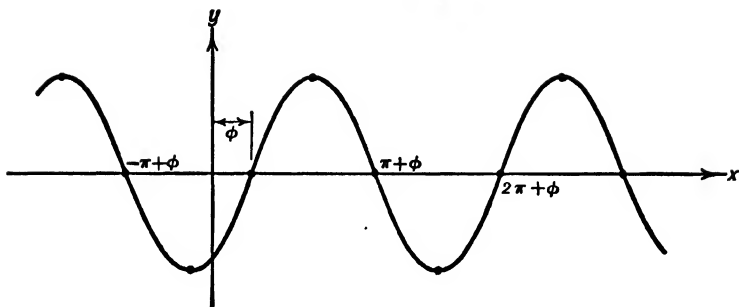


FIG. 123

The curve is drawn in Figure 123. It is identical with Figure 120, except that the entire curve has been shifted to the right, through a distance ϕ . The quantity ϕ is called the *displacement in phase* of the sine wave.

Exercises

Sketch the curves represented by each of the following eight equations, choosing appropriate scales.

1. $y = 3 \sin 5x$

2. $y = 10 \sin (2x + \pi/2)$

3. $y = 10 \cos 2x$

4. $y = 170 \cos 120\pi x$

5. $y = 4 \sin \left(\frac{3x}{2} - \frac{\pi}{6} \right)$

6. $y = 6 \cos \left(\frac{x}{2} - \frac{\pi}{4} \right)$

7. $y = A(1 - \sin x)$

8. $y = A(1 - \cos x)$

9. Show that $3 \sin x + 4 \cos x = 5 \sin (x + \phi)$, where $\tan \phi = \frac{4}{3}$. Sketch the curve $y = 3 \sin x + 4 \cos x$.

10. Show that

$$A \sin x + B \cos x = \sqrt{A^2 + B^2} \sin (x + \phi)$$

where $\tan \phi = B/A$. What conclusion can be drawn about the nature of any curve represented by an equation of the form $y = A \sin x + B \cos x$?

CHAPTER 20

THREE-DIMENSIONAL GEOMETRY

134. Introduction. In the development of an analytic theory of the geometry of three-dimensional configurations, it is natural to require that the new theory contain the geometry of a plane as a special case. This suggests that we proceed by extending our results in plane geometry to the three-dimensional case, step by step. Once again a motif that penetrates and informs the whole range of mathematical thought, *the urge to generalize*, directs our steps.

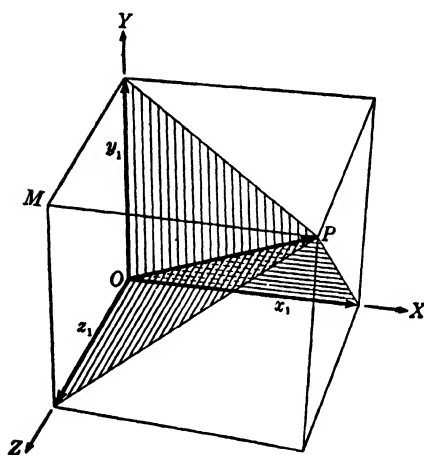


FIG. 124

trates and informs the whole range of mathematical thought, *the urge to generalize*, directs our steps.

135. Coordinates in Space. The location of a point in space is usually given with reference to three coordinate planes, determined by three mutually perpendicular axes (Figure 124). Imagine a box, so placed that one corner is at the origin of axes, and the corner diagonally opposite is at the point $P(x_1, y_1, z_1)$. The box is oriented as in Figure 124. The meaning of the coordinates x_1 , y_1 , and z_1 , can now be expressed in several equivalent ways (a) The coordinates x_1 , y_1 , and z_1 represent

the length, height, and breadth of the box. (b) The coordinate x_1 represents the distance of P from the coordinate plane yz . This is the length PM in Figure 124. Similarly, y_1 represents the height of P above the xz -plane, and z_1 represents the distance from the xy -plane. (c) The point P determines a line vector OP , running from the origin to P . The components of the vector, resolved along the three coordinate axes, are x_1 , y_1 , z_1 .

In plotting points, it is convenient to use quadrille-ruled paper, drawing the z -axis at an angle of 45° . One diagonal unit along the z -axis may be reckoned as equivalent to two units along either the x -axis or y -axis. The resulting figure will be faulty in perspective, but for many purposes

a moderate degree of distortion is permissible. The illustrations in the text have been drawn in perspective, to facilitate the visualization of geometric relationships.

Exercises

1. Plot the points $(1,1,1)$, $(5,0,0)$, $(0,0,-2)$, $(-3,0,2)$, $(2,6,1)$.
2. What are the coordinates of the point at the foot of the perpendicular from $P(4,2,3)$ to the xy -plane? From P to the yz -plane?
3. What are the coordinates of the point at the foot of the perpendicular from $P(a,b,c)$ to the x -axis? From P to the y -axis?
4. What is the locus of points for which $x = 2$? Of points for which $z = -3$? Find an equation representing the plane consisting of points at a height of 6 units above the xz -plane.
5. Find the distance from the point $P(x_1, y_1, z_1)$ to the x -axis. From P to the z -axis. From P to the origin.
6. Find an equation representing the yz -plane.
7. What is the locus of all points satisfying the equation $xyz = 0$?

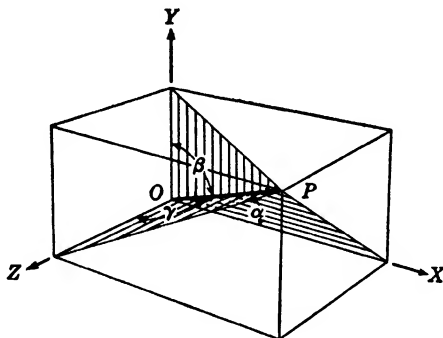


FIG. 125

136. Direction Cosines. The vector OP running from the origin to the point $P(x_1, y_1, z_1)$ is completely specified by its components x_1 , y_1 , and z_1 . It may also be specified by its length and direction. Direction may be indicated by the *direction angles* α , β , γ (Figure 125). The angle α lies in a plane passing through OP and the x -axis. It is measured from the positive x -axis to OP , and always lies between 0° and 180° . The second direction angle, β , lies in a plane passing through OP and the y -axis. The third direction angle, γ , lies in a plane passing through OP and the z -axis.

Let r represent the length of OP . Then

$$\begin{aligned} r^2 &= (OM)^2 + (MP)^2 \\ &= (OM)^2 + (PN)^2 + (NM)^2 \\ r^2 &= x_1^2 + y_1^2 + z_1^2 \end{aligned} \quad 20-1$$

Since x_1 , y_1 , and z_1 are the projections of the line vector OP upon the three coordinate axes, we have, from the definition of the cosine (equation 1-1),

$$\begin{aligned} x_1 &= r \cos \alpha \\ y_1 &= r \cos \beta \\ z_1 &= r \cos \gamma \end{aligned} \quad 20-2$$

From these equations we see that the coordinates of a point are proportional to the direction cosines of a line vector running from the origin to the point.

The direction angles of a line are not independent of one another. If any two are specified, the third may be calculated by means of the equation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad 20-3$$

The verification of formula 20-3 is left as an exercise for the student.

Exercises

1. Find the direction angles of the vectors running from the origin to the points $(1,1,1)$, $(2,-3,-3)$, and $(-4,2,-1)$.
2. By taking $z_1 = 0$, obtain the formula for the slope of a line running from the origin to any point in the xy -plane as a special case of the formulas 20-2.
3. What are the direction cosines of the x -axis? Of the y -axis?
4. Verify equation 20-3 by means of equations 20-1 and 20-2.
5. What are the direction cosines of a line making equal angles with the coordinate axes?
6. Show that, if we restrict ourselves to points in the xy -plane, equation 20-3 reduces to the familiar identity

$$\sin^2 A + \cos^2 A = 1$$

137. The Distance Between Two Points. Through a point $P_1(x_1, y_1, z_1)$, imagine three planes to be passed, parallel to the coordinate planes. Similarly, let three planes be passed through a second point $P_2(x_2, y_2, z_2)$. These six planes form the boundaries of a box-shaped figure (Figure 126).

The distance of P_1 from the yz -plane is x_1 . The distance of P_2 from the same plane is x_2 . Hence the length of the box, parallel to the x -axis, is $x_2 - x_1$. That is, $x_2 - x_1$ is the component of the line vector P_1P_2 parallel to the x -axis.

The coordinate y_1 represents the height of P_1 above the xz -plane, and y_2 represents the height of P_2 . Hence the height of the box is $y_2 - y_1$. That is, $y_2 - y_1$ is the component of P_1P_2 parallel to the y -axis.

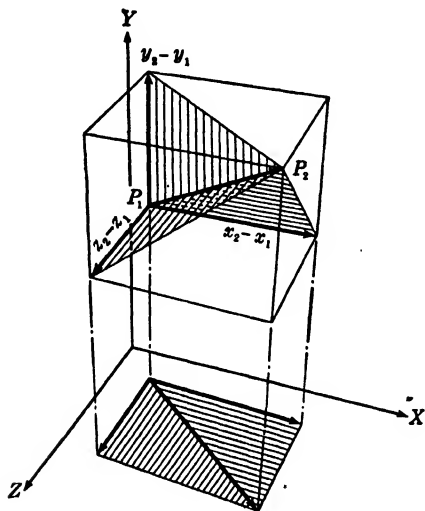


FIG. 126

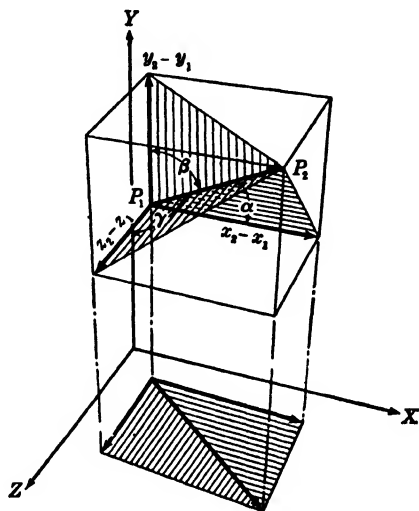


FIG. 127

In the same manner, it is easily seen that $z_2 - z_1$ is the breadth of the box. That is, $z_2 - z_1$ is the component of P_1P_2 parallel to the z -axis. These results should be compared with the two-dimensional formulas on page 179.

The distance P_1P_2 is expressed by the formula

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad 20-4$$

The direction angles of P_1P_2 are defined as the direction angles of a vector, parallel to P_1P_2 , passing through the origin of axes. They are shown in Figure 127, from which it can be seen that the component of a line vector along the x -axis is obtained by multiplying the length of the vector by the cosine of the first direction angle. The other two components are obtained in a similar manner. These results are expressed by the formulas

$$x_2 - x_1 = P_1P_2 \cos \alpha$$

$$y_2 - y_1 = P_1P_2 \cos \beta \quad 20-5$$

$$z_2 - z_1 = P_1P_2 \cos \gamma$$

These formulas may be used to calculate the direction cosines, if the coordinates of P_1 and P_2 are known. They are also useful in calculating the components of the vector quantities occurring in various branches of applied mathematics.

Exercises

1. Find the direction cosines of the line running from $(-1, 3, 2)$ to $(4, 1, -5)$.
2. The *direction numbers* of a line are defined to be any set of numbers proportional to the direction cosines. Find two such sets for the line of Exercise 1. Is it true that the components of a line vector always furnish one set of direction numbers for a line in the direction of the vector?
3. Complete the following statement: The components of a vector one unit long are numerically equal to —.
4. Find α , β , and γ for a line running from $(2, 4, -1)$ to $(-3, 3, 2)$.
5. Find the direction angles of a line running from $(-2, 2, 5)$ to $(1, 0, 1)$.
6. Find the components of a force of 240 lb., whose line of action runs from $(0, 1, 0)$ to $(-2, 3, 1)$.
7. Find the components of a force of 350 lb., whose line of action runs from $(-3, 1, 2)$ to $(4, -5, 5)$.
8. Show that the three points $(-4, 2, 7)$, $(2, -1, 4)$, and $(10, -5, 0)$ lie on a straight line.
9. Complete the following statement: The direction cosines of a line determined by two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to —.
10. What is the effect of reversing a line vector upon the components of the vector? What are the direction cosines of the line vector running from P_2 to P_1 ?

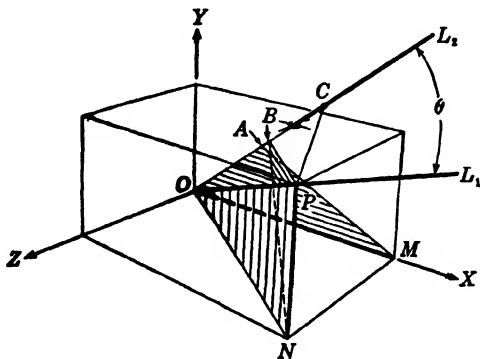


FIG. 128

138. The Angle Between Two Lines. Two lines having one point in common are called *concurrent*. Figure 128 shows two lines, L_1 and L_2 , which are concurrent at the origin. OP is a vector along L_1 , whose three rectangular components are OM , MN , and NP . The points A , B , C are

obtained by dropping perpendiculars from points M , N , and P upon L_2 . Hence

OC is the projection of OP upon L_2 ;

OA is the projection of OM upon L_2 ;

AB is the projection of MN upon L_2 ;

BC is the projection of NP upon L_2 .

Now the projection of a vector on any line is equal to the length of the vector multiplied by the cosine of the included angle. Thus

$$OC = OP \cos \theta$$

$$OA = OM \cos \alpha_2$$

$$AB = MN \cos \beta_2$$

$$BC = NP \cos \gamma_2$$

where $\alpha_2, \beta_2, \gamma_2$ are the direction angles of L_2 . But it is evident from the figure that

$$OC = OA + AB + BC$$

$$\therefore OP \cos \theta = OM \cos \alpha_2 + MN \cos \beta_2 + NP \cos \gamma_2$$

We next observe that

$$OM = OP \cos \alpha_1$$

$$MN = OP \cos \beta_1$$

$$NP = OP \cos \gamma_1$$

where $\alpha_1, \beta_1, \gamma_1$ are the direction angles of L_1 . Hence we have the equation

$$\cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2 \quad 20-6$$

The angle between any two lines in space is defined to be the angle between two concurrent lines, parallel to the given lines. We have therefore showed that *the cosine of the angle between two lines is equal to the sum of the products of the corresponding direction cosines of the lines.*

Exercises

1. Write a formula expressing the relation that must be satisfied by the direction cosines of two perpendicular lines.
2. Derive the fundamental relation of analytical trigonometry (equation 10-1) as a special case of equation 20-6, by restricting the given lines to the xy -plane.
3. Find the angles of the triangle formed by joining the points $(0,1,2)$, $(3,0,0)$, and $(0,5,0)$.
4. Find the angles of the triangle formed by joining the points $(0,0,4)$, $(6,0,4)$, and $(0,3,0)$.

5. Find the direction angles of a line in the xy -plane, whose equation in that plane is $2x + 3y = 1$.
6. Find the direction angles of a line in the xy -plane, whose equation in that plane is $x - 2y = 3$.
7. Find the angle between the lines of Exercises 5 and 6.

139. The Equation of a Plane. In the geometry of two dimensions, the locus of all points satisfying an equation in x and y , is a curve (which, of course, may have more than one branch).

In the geometry of space, the locus of all points satisfying an equation in x , y , and z , is a *surface*. An equation is said to *represent* a surface if

the coordinates of every point on the surface satisfy the equation, and if every set of values of x , y , and z that satisfy the equation are the coordinates of a point on the surface.

In order to find the equation that represents a surface, we look for a suitable geometric condition defining the surface, and then express this geometric condition by an equation. For example, a plane may be defined by the condition that a certain line, called the *normal* to the plane,

is perpendicular to every line in the plane through its foot. Let the direction of the normal be that of a vector having the components A , B , and C along the coordinate axes, so that the direction cosines of the normal are

$$\cos \alpha_1 = \frac{A}{\sqrt{A^2 + B^2 + C^2}}$$

$$\cos \beta_1 = \frac{B}{\sqrt{A^2 + B^2 + C^2}}$$

20-7

$$\cos \gamma_1 = \frac{C}{\sqrt{A^2 + B^2 + C^2}}$$

Let $P(x_1, y_1, z_1)$ be a known point on the plane, and let $Q(x, y, z)$ be any other point on the plane (Figure 129). Then the line joining P and Q , with direction angles α_2 , β_2 , and γ_2 , is perpendicular to the normal at P . This statement is true for every point Q lying on the plane. An equation expressing this condition is obtained from equation 20-6. Taking $\theta = 90^\circ$,

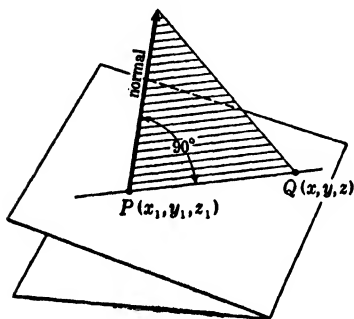


FIG. 129

we have

$$\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2 = 0$$

The direction cosines of the normal are given by equations 20-7. The direction cosines of PQ may be found by equations 20-5. Accordingly, we may write the equation

$$\frac{A}{R} \left(\frac{x - x_1}{PQ} \right) + \frac{B}{R} \left(\frac{y - y_1}{PQ} \right) + \frac{C}{R} \left(\frac{z - z_1}{PQ} \right) = 0$$

where R stands for the radical $\sqrt{A^2 + B^2 + C^2}$. This equation is satisfied if, and only if, $Q(x, y, z)$ lies on the plane. If both members are multiplied by the common denominator, it takes the form

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \quad 20-8$$

By replacing the quantity $-Ax_1 - By_1 - Cz_1$ with a single constant D , we obtain

$$Ax + By + Cz + D = 0 \quad 20-9$$

Thus we see that every equation of the first degree represents a plane in space, the direction of the plane being given by a normal vector whose components are the coefficients of x , y , and z .

Example 1. Find the equation of the plane passing through the points $(0, 2, -3)$, $(-1, 0, 5)$, and $(1, 1, -2)$.

In equation 20-9, there are but three independent constants. We may therefore assign any arbitrary value, other than zero, to one of the four constants, with no loss in generality. Accordingly, let us set $D = 1$. Since each of the given points lies on the plane, its coordinates satisfy the equation of the plane. We thus obtain the simultaneous equations

$$2B - 3C + 1 = 0$$

$$-A + 5C + 1 = 0$$

$$A + B - 2C + 1 = 0$$

Upon solving, we find $A = -\frac{2}{3}$, $B = -1$, and $C = -\frac{1}{3}$. The equation of the plane is

$$-\frac{2}{3}x - y - \frac{1}{3}z + 1 = 0$$

Or

$$2x + 3y + z = 3$$

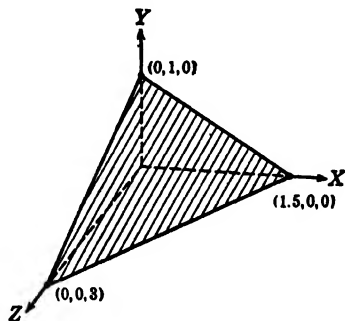


FIG. 130

The plane is readily sketched by finding its intercepts on the coordinate axes. When $y = z = 0$, $x = \frac{3}{2}$. When $x = z = 0$, $y = 1$. When $x = y = 0$, $z = 3$. The plane is shown in Figure 130.

Example 2. Find the direction cosines of the normal to the plane whose equation is

$$2x - y + 2z + 5 = 0$$

The normal vector has components 2, -1, and 2. Its length is therefore

$$\sqrt{2^2 + (-1)^2 + 2^2} = 3$$

Hence we may write

$$\cos \alpha = \frac{2}{3}$$

$$\cos \beta = -\frac{1}{3}$$

$$\cos \gamma = \frac{2}{3}$$

Exercises

1. Write the equation of a plane, passing through the point (1,3,2), parallel to the yz -plane. What are the direction cosines of the normal?
2. A plane passes through the point (1,3,2), and is parallel to the xz -plane. Find its equation. What are the direction cosines of the normal?
3. Find the direction cosines of the normal to the plane

$$x - 2y + z = 3$$

4. Find the direction cosines of the normal to the plane

$$2x + y - 3z = 1$$

5. Find the angle between the normals to the planes of Exercises 3 and 4, and show that this angle equals the angle between the planes.
6. Find the angle between the planes

$$4x - 3z = 0$$

$$x + 2y - z = 2$$

7. Find the points at which the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

intersects the coordinate axes, and draw the plane.

8. Show that the plane

$$4x + 2y - 6z = 3$$

is parallel to the plane of Exercise 4.

9. Find the equation of the plane determined by the points (2, -1, 5), (3, 2, 6), and (1, 1, -5).

140. The Equations of a Straight Line. Any two non-parallel planes will intersect in a straight line. This line is the locus of all points satisfying the equations of both planes simultaneously. Thus any two inde-

pendent equations of the first degree, taken simultaneously, may be regarded as representing a straight line in space.

That our present view is consistent with the treatment of the straight line in two dimensions is easily shown; for the equation $Ax + By + D = 0$ in two dimensions may be regarded as the intersection of the plane $Ax + By + Cz + D = 0$ with the xy -plane $z = 0$.

Let us find equations representing a straight line, passing through a given point $P_1(x_1, y_1, z_1)$, and having the direction angles α , β , and γ . If $P(x, y, z)$ is any other point on the line, we have from equations 20-5

$$x - x_1 = P_1P \cos \alpha$$

$$y - y_1 = P_1P \cos \beta$$

$$z - z_1 = P_1P \cos \gamma$$

Eliminating the distance P_1P , we obtain

$$\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma}$$

If A , B , C are the components of a vector having the direction of the line, these components are proportional to the direction cosines. Hence we may take the equations of the line in the *symmetric form*

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C} \quad 20-10$$

Example. A line is determined by the equations

$$x - 2y + z = 2$$

$$2x + y - 3z = 4$$

It is required to draw the line, and to find its equations in the symmetric form.

We first locate the *piercing point* in which the line intersects the xy -plane. Set $z = 0$, and solve the two equations simultaneously. In this way, we find $(2, 0, 0)$ to be one point on the line.

Next, we locate a second point, where the line pierces the yz -plane. Setting $x = 0$, we find $(0, -2, -2)$ to be another point on the line. It is now easy to sketch the line, since two points on it are known. The components of the line vector joining the two known points are 2, 2, 2. Hence we may write the equations of the line in the form

$$\frac{x}{2} = \frac{y + 2}{2} = \frac{z + 2}{2}$$

Exercises

1. Find a pair of equations of the line through the points $(-1, 3, 3)$ and $(2, -5, 4)$.
2. Find a pair of equations of the line through the points $(2, -1, 0)$ and $(5, -1, -3)$.
3. Find a pair of equations of the line through the origin, perpendicular to the plane $x - 2y + 3z = 5$.
4. Find a pair of equations of the line through the point $(3, 1, 1)$, perpendicular to the plane $3x + 2y - z = 1$.
5. Find the direction cosines of the line whose equations are $6x - 2y + z = 2$ and $2x + 4y - 3z = 1$, and sketch the line.
6. Find the direction cosines of the line whose equations are $2x - y + 3z = 5$ and $4x + y - 3z = 3$, and sketch the line.
7. Find the angle between the lines of Exercises 5 and 6.
8. Show that the lines

$$x + 5 = 3y, \quad 4y = z + 9$$

and

$$x + z = 0, \quad x + 1 = y$$

are concurrent and perpendicular.

141. Cylindrical Surfaces. A cylindrical surface is generated by a straight line, moving in such a manner as to remain always parallel to a fixed line. If the generating line is parallel to the x -axis, the equation of the surface does not contain x . The converse statement is also true; if the equation contains only y and z , the locus in space is a cylinder with elements parallel to the x -axis. Similar statements may be made with reference to cylinders whose generating lines are parallel to either of the other coordinate axes. The reasoning, upon which these theorems are based, is illustrated in the following examples.

Example 1. Draw the surface whose equation is

$$x^2 + z^2 = 25$$

In the xz -plane, this equation represents a circle of radius 5, with center at the origin. Any point on the circle satisfies the equation of the circle. Next, consider the points on a straight line, perpendicular to the plane of the circle, and containing one point of the circle. All points on this line have the same x -coordinate, and the same z -coordinate; only the y -coordinate varies from one point to another. Hence the coordinates of any point on the line will satisfy the equation of the circle. Thus the locus is a *circular cylinder*, whose axis is the y -axis, and whose radius is 5. A part of this infinite cylinder is shown in Figure 131.

Example 2. Sketch the locus represented by the equation

$$2y + z^2 = 4$$

In the yz -plane, this equation represents a parabola, symmetrical about the y -axis. The vertex of the parabola is at $(0,2,0)$, and the z -intercepts are ± 2 . Reasoning as before, we see that the coordinates of every point on a line, parallel

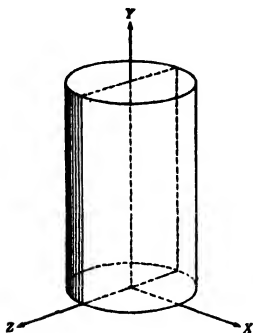


FIG. 131

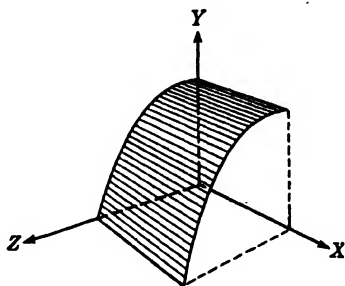


FIG. 132

to the x -axis, and containing a point on the parabola, will satisfy the given equation. Hence the locus is a *parabolic cylinder*, part of which is shown in Figure 132.

Exercises

Identify and sketch the following eight surfaces.

1. $x^2 + y^2 = 4y$

2. $x^2 + 4y^2 = 4x$

3. $z^2 = 9x$

4. $z^2 = 2y + 4$

5. $x^2 - z^2 = 16$

6. $xz = 4$

7. $y = 4 \cos \pi x$

8. $y = 3 \sin \pi x/2$

9. Find the equation of a right circular cylinder of radius r , whose axis is parallel to the x -axis, and passes through the point $(0, r, 0)$.
 10. Find an equation representing an elliptic cylinder, whose axis is the y -axis.

142. Surfaces of Revolution. A surface generated by revolving a curve about a straight line is called a surface of revolution. Every plane, perpendicular to the straight line, cuts the surface in a circle.

Let $F(x, y, z) = 0$ be the equation of a surface of revolution, whose axis is the x -axis. Let the surface be cut by a plane perpendicular to the x -axis, as shown in Figure 133. Every point on the circle of intersection must satisfy the equation

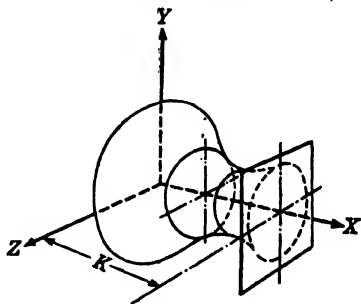


FIG. 133

of the surface, and also the equation of the plane, namely $x = K$. But we know that the equation of the circle must have the form

$$y^2 + z^2 = \text{constant}$$

Hence the equation of the surface must reduce to this form when x is replaced by K .

Similar reasoning shows that, if the axis of revolution is the y -axis, the equation of the surface must reduce to the form

$$x^2 + z^2 = \text{constant}$$

when y is replaced by K .

If the axis of revolution is the z -axis, the equation of the surface must reduce to the form

$$x^2 + y^2 = \text{constant}$$

when z is replaced by K .

Example 1. Sketch the surface whose equation is

$$x^2 + y^2 = 4z$$

When $z = K$, the equation has the form

$$x^2 + y^2 = \text{constant}$$

Hence it represents a surface of revolution. The z -axis is the axis of revolution.

The curve, which is the intersection of the surface with the xz -plane, can be obtained by setting $y = 0$. The equation of this curve is

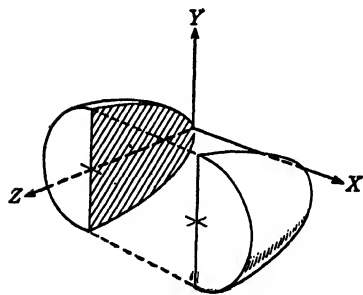


FIG. 134

$$x^2 = 4z$$

which is a parabola, whose axis is the z -axis. The surface under investigation can be generated by revolving the parabolic curve $x^2 = 4z$ about its own axis. The surface is called a *paraboloid of revolution* (Figure 134).

Example 2. Find the equation of the *ellipsoid of revolution* generated by revolving about its major axis the ellipse whose equation in the xy -plane is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Let $P(x, y, 0)$ be any point on the ellipse. Upon rotating the ellipse about the x -axis, the point P describes a circle, whose radius is equal to the initial value of y . Thus, for any point on the circle generated by P , the quantity $\sqrt{y^2 + z^2}$

must have the value of the y -coordinate in the initial position. Hence the equation

$$\frac{x^2}{25} + \frac{y^2 + z^2}{9} = 1$$

represents the surface of revolution under discussion.

In general, if $f(x, y) = 0$ is the equation of any curve in the xy -plane, then $f(x, \sqrt{y^2 + z^2}) = 0$ is the equation of the surface obtained by rotating the given curve about the x -axis. That is, if the letter y in the equation of the curve be replaced by $\sqrt{y^2 + z^2}$, the new equation represents the surface generated by revolving the curve about the x -axis.

The foregoing statement evidently remains valid if any two of the letters x, y, z are interchanged.

Exercises

1. How would you obtain the equation of a surface generated by revolving a curve in the xz -plane about the x -axis? Express your result in the f -notation.
2. How would you obtain the equation of a surface generated by revolving a curve in the yz -plane about the z -axis? Express your result in the f -notation.

Identify and sketch the surfaces represented by each of the following eight equations.

- | | |
|------------------------------|----------------------------|
| 3. $4x^2 + 4y^2 + 4z^2 = 25$ | 4. $x^2 + y^2 + z^2 = 8z$ |
| 5. $x^2 = y^2 + z^2$ | 6. $x^2 + 4y^2 + 4z^2 = 4$ |
| 7. $x = 4y^2 + 4z^2$ | 8. $4x^2 + 4y^2 = z^2 + 4$ |
| 9. $4x^2 = y^2 + z^2 + 4$ | 10. $x^2 + y^2 = z^4$ |

11. Find the equation of a sphere with center at the origin, and radius r .
12. Find the equation of the surface generated by rotating the hyperbola

$$x^2 - y^2 = a^2$$

about the y -axis. (This surface is called a hyperboloid of one sheet.)

13. Find the equation of the surface generated by rotating the hyperbola

$$x^2 - y^2 = a^2$$

about its own axis. (This surface is called a hyperboloid of two sheets.)

14. Find the equation of the torus generated by revolving about the x -axis a circle of radius 3, lying in the xy -plane, with center at $(0, 5, 0)$.
15. Find the equation of the cone whose axis is the y -axis, and whose vertex angle is 2ϕ

143. Quadric Surfaces. We next consider some examples of surfaces whose equations have the form

$$Ax^2 + By^2 + Cz^2 = 1 \qquad 20-11$$

If any of the constants A, B, C vanish, the equation represents a cylindrical surface. If any two of the coefficients are equal, the surface is one of revolution.

Case I. If A , B , and C are all positive, the surface represents an *ellipsoid*. Every plane section of the surface is an ellipse.

Case II. If just one coefficient is negative, the surface represents a *hyperboloid of one sheet*. This resembles the hyperboloid of revolution, generated by rotating a hyperbola about its conjugate axis; but in the general case, plane sections taken perpendicular to the conjugate axis are ellipses, instead of circles.

Case III. If just two coefficients are negative, the surface represents a *hyperboloid of two sheets*. This resembles the surface generated by rotating a hyperbola about its own axis, except that plane sections taken perpendicular to the axis of the hyperbola are ellipses, instead of circles.

Example 1. Consider the surface

$$4x^2 + 16y^2 + 9z^2 = 144$$

The xy -plane intersects the surface in a curve whose equation is $4x^2 + 16y^2 = 144$. But this is an ellipse, whose major semi-axis is 6, and minor semi-axis 3.

The xz -plane intersects the surface in the ellipse $4x^2 + 9z^2 = 144$. The yz -plane intersects the surface in the ellipse $16y^2 + 9z^2 = 144$.

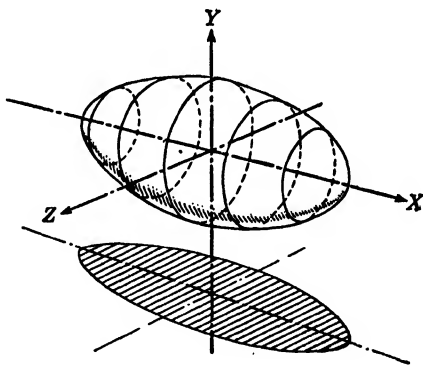


FIG. 135

Every plane parallel to the xy -plane has an equation of the form $z = k$. The points lying on the curve, in which this plane intersects the surface, must satisfy the equations of both. Hence the equation of the curve of intersection is

$$4x^2 + 16y^2 = 144 - 9k^2 = \text{constant}$$

It is evident that all plane sections, taken parallel to the coordinate planes, are ellipses. Moreover, by applying the transformation theory of Chapter 18, it is not difficult to show that *every plane section of this surface is an ellipse*. The surface is drawn in Figure 135.

Example 2. Consider the surface

$$8x^2 - 2y^2 + 16z^2 = 32$$

All plane sections perpendicular to the y -axis have equations of the form

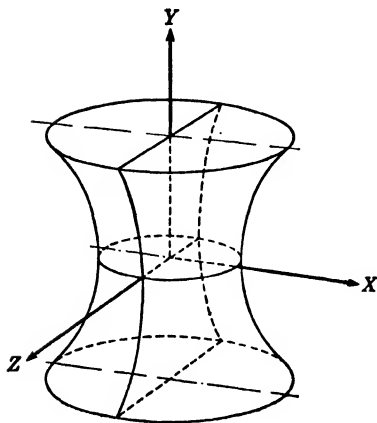


FIG. 136

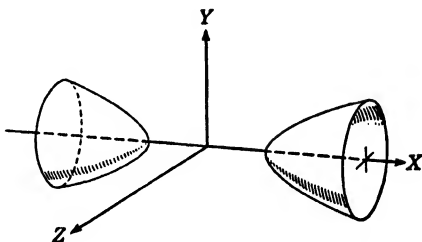


FIG. 137

$8x^2 + 16z^2 = \text{constant}$. These are ellipses. Plane sections perpendicular to the other coordinate axes are hyperbolas. The surface is drawn in Figure 136.

Example 3. Consider the surface

$$4x^2 - 9y^2 - 36z^2 = 36$$

The plane $x = k$ intersects the surface in a curve whose equation is $9y^2 + 36z^2 = 4k^2 - 36$. The intersection is imaginary if k^2 is less than 9. For larger values, the curve of intersection is an ellipse. Plane sections perpendicular to the other two coordinate axes are hyperbolas. The surface is drawn in Figure 137.

Exercises

Identify the surfaces represented by the following five equations, and sketch each.

1. $y^2 + 4z^2 = 16 + x^2$
2. $9x^2 + y^2 + 18z^2 = 9$
3. $12y^2 = 4x^2 + 3z^2 + 12$
4. $5x^2 + 20y^2 = 10z^2 + 20$
5. $36z^2 = 36 + 9x^2 + 4y^2$

6. Study the plane sections of the *elliptic paraboloid*

$$x^2 + 4z^2 = 4y$$

and sketch the surface.

Miscellaneous Problems

1. Solve the simultaneous equations

$$\begin{cases} 4x^2 + 4y^2 = 65 \\ xy + 2 = 0 \end{cases}$$

by algebra, and check by drawing both curves on the same pair of coordinate axes.

2. A point moves so that its distance from (0,5) is always five-thirds of its distance from the line $5y - 9 = 0$. Find the equation of its path in simplest form, and draw the curve.
3. Reduce the equation

$$4y^2 - 36x - 12y = 57$$

to standard form, and draw the curve.

4. Reduce the equation

$$25x^2 + 150x = 8y^2 - 25$$

to standard form, and draw the curve.

5. Change the equation

$$2r - 4r \cos \theta = 3$$

to rectangular coordinates, and draw the curve.

6. What does the equation of the curve

$$xy + 2 = 0$$

become when the x -axis is moved 4 units downward, and the y -axis is moved 1 unit to the right?

7. Draw the locus whose equation is

$$(x^2 - 2y)(x^2 - 2y^2) = 0$$

9. A straight line passes through the points whose polar coordinates are $(\sqrt{8}, -3\pi/4)$ and $(5, \pi)$. Find its equation in rectangular coordinates.
10. The towers supporting a suspension bridge are 300 feet apart and rise 86 feet above the road-bed. The lowest point of the parabola formed by the main cables is 12 feet above the road-bed. Find the length of vertical cables, reaching from the main cables to the floor of the bridge, at intervals of 30 feet from the center to either end.
11. Find the equation of the parabola with axis parallel to the x -axis, passing through the points $(-2, 3)$, $(10, 0)$, and $(4, -3)$.
12. Find the equation of the plane which is tangent to the sphere

$$x^2 + y^2 + z^2 - 2x + 4y = 31$$

at the point $(3, 2, -4)$.

13. Remove the term in
- xy
- from the equation

$$2x + 3y + \sqrt{2}xy + \sqrt{2} = 0$$

by a rotation of axes, and draw the curve.

14. An ellipsoid of revolution has its center at the origin and its longest axis along the x -axis. If the length of the longest axis is 8, and the ellipsoid passes through the point $(2,0,3)$, what is its equation?
15. Draw the solid figure which is bounded by the surfaces $y^2 = x$, $2x + y = 2$, $z = 0$, and $z = 3$.
16. Write the equation of the ellipse with center at $(1,2)$, one vertex at $(5,2)$, and one focus at $(3,2)$.
17. Draw the curve $y = 4 \sin (2x + \pi)$.
18. A line in the yz plane passes through the points whose coordinates in that plane are $(2, -1)$ and $(-3, 5)$. What are the direction cosines of this line in space?
19. Find the angle between the line of Problem 18 and the normal to the plane $2x + 3y - z = 4$.
20. Draw the curve

$$r^2 (2 + \sin 2\theta) = 10$$

21. Find the equation of the ellipse which has the line segment joining $(-3, 1)$ and $(5, 1)$ as its major axis, and which passes through the point $(3, 4)$. What is the eccentricity of this ellipse?

CHAPTER 21

DERIVATIVES AND INTEGRALS

144. Motion. It happens not infrequently that familiar phenomena present at first a specious air of simplicity, but when critically examined they are found to offer deep and subtle obstacles to scientific inquiry. The attempt to explain motion has attracted the best minds of the race for more than two thousand years. The fundamental problem of motion is to find simple and general mathematical laws from which, given the present state of a physical system, the position of any desired object in the system can be predicted for any time in the reasonably foreseeable future.

A difficulty encountered by the student at the outset is that of acquiring clear ideas about what is meant by *velocity* and *acceleration*; for vague and confused notions, arising from common experience of these things, do not provide a satisfactory foundation for engineering and scientific work.

145. Average Velocity. When a body moves in a straight line, in such a way that equal distances are covered in equal times, the motion is said to be *uniform*, and the velocity is defined to be the distance covered in unit time. In most engineering work, velocity is expressed in feet per second. For uniform motion, the distance from the starting point at any time is expressed by the formula

$$s = kt$$

where the constant k represents the velocity.

Let us next consider an example of non-uniform motion. Suppose that the distance (in feet) of a moving car from its starting point is expressed by the formula

$$s = 3t^2 \qquad 21-1$$

The average velocity during any period of time is defined to be the *change in s* divided by the *time during which the change occurs*. We shall use the special symbol Δt to represent the time interval. The corresponding symbol Δs represents the change in s . These symbols are read *delta t* and *delta s*. They should be regarded as single quantities, even though the symbol is composite. The average velocity is expressed in the delta

notation by the formula

$$\text{ave. vel.} = \frac{\Delta s}{\Delta t} \quad 21-2$$

Let us calculate the average velocity of the moving car during the fifth second of its motion, that is, from $t = 4$ to $t = 5$. In this case, the time interval is

$$\Delta t = 5 - 4 = 1 \text{ second}$$

From equation 21-1, the corresponding distances are 48 and 75 feet from the starting point. Hence the change in position is

$$\Delta s = 75 - 48 = 27 \text{ feet}$$

The average velocity is

$$\frac{\Delta s}{\Delta t} = 27 \text{ feet per second}$$

Let us next obtain a formula for the average velocity during any interval Δt , beginning at the time $t = 4$. From equation 21-1, the value of s at the time $4 + \Delta t$ is

$$3(4 + \Delta t)^2 = 48 + 24\Delta t + 3(\Delta t)^2$$

Hence the change in distance is

$$\Delta s = 24\Delta t + 3(\Delta t)^2$$

Dividing both members by Δt , we have the desired formula

$$\frac{\Delta s}{\Delta t} = 24 + 3\Delta t \quad 21-3$$

This result greatly facilitates the calculation of average velocities during any interval, positive or negative, beginning with $t = 4$. For example, during the first third of the fifth second, the average velocity is 25. During the preceding one-tenth of a second, the average velocity is 23.7 (taking $\Delta t = -0.1$).

By an easy extension of the above method, we may obtain a formula for the average velocity during any time interval whatever. Let t be the time at the beginning of the interval. If Δt is the length of the time interval, the time at the end of the interval will be $t + \Delta t$. In like manner, we write s for the distance from the starting point at the beginning of the time interval, Δs for the change in distance, and $s + \Delta s$ for the final dis-

tance. From equation 21-1, it follows that

$$\begin{aligned}s + \Delta s &= 3(t + \Delta t)^2 \\ &= 3t^2 + 6t \Delta t + 3(\Delta t)^2\end{aligned}$$

Subtracting $s = 3t^2$, the result is

$$\Delta s = 6t \Delta t + 3(\Delta t)^2$$

Dividing both members by Δt , we have the formula

$$\frac{\Delta s}{\Delta t} = 6t + 3\Delta t \quad 21-4$$

for the average velocity during the interval from t to $t + \Delta t$.

Exercises

1. Calculate the average velocity of the car discussed in the text during the seventh second.
2. Calculate the average velocity of the car from $t = 5.8$ to $t = 6$.
3. The height of an elevator at any time is given by the equation

$$h = 5t^2 + 12$$

Find a formula for the average velocity during any interval. What is the average velocity during the sixth second?

4. The height of a bomb is given by the formula

$$h = -16.1t^2 + 5500$$

Find a formula for the average velocity of the bomb during any interval. How long does it take the bomb to reach the ground? What was the height at starting? What is the average velocity during the fall?

5. If s is plotted against t for uniform motion, show that a straight line results. What physical quantity is represented by the slope of this line? (Note that s may be measured from any fixed point, not necessarily the starting point.)
6. Express the slope formula 16-2 in the delta notation.

146. Instantaneous Velocity. When the distance traversed by a moving object is given by the formula $s = 3t^2$, the average velocity during any time interval beginning at $t = 4$ has been found to be

$$\frac{\Delta s}{\Delta t} = 24 + 3\Delta t \quad 21-3$$

If the time interval Δt is very short, the average velocity differs but little from the value 24 feet per second. This value is called the instantaneous velocity at $t = 4$. Henceforth, the adjective *instantaneous* will be omitted, for the sake of brevity.

The velocity at any moment is defined to be the limit approached by the average velocity, during a time interval including the moment in question, as the interval is taken shorter and shorter. In symbolic language, velocity at an instant is defined to be

$$\frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} \quad 21-5$$

In the example of the moving car, the velocity at any instant is obtained from formula 21-4, by finding the limiting value of the average velocity, when Δt approaches zero. We see that

$$\frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} (6t + 3\Delta t) = 6t$$

Thus, at the moment $t = 11$, the velocity is 66 feet per second.

From the physical point of view the possibility of making measurements, involving a time interval that is arbitrarily small, may well be questioned. However, it must be remembered that *no physical phenomenon is described exactly by a mathematical formula*. For the engineer, a mathematical formula is satisfactory if it represents the physical problem with sufficient accuracy for the purpose in hand. If several formulas are available which satisfy this criterion, then the simplest formula is the best. Thus one reason for the use of instantaneous velocities is found in the fact that many formulas for instantaneous velocities are simpler than the corresponding formulas for average velocities.

The evaluation of the limit in equation 21-5 is not always an easy matter, and the practical utility of instantaneous velocities would be greatly lessened if it were necessary to evaluate this limit directly in all cases. In the next section, we shall see that it is possible to find velocities by rule.

Example 1. The position of a moving object is expressed by the equation

$$s = \frac{2 - t}{t}$$

It is required to find the velocity at any instant.

The first step is to allow t to increase by an amount Δt . The new position is given by the equation

$$s + \Delta s = \frac{2 - (t + \Delta t)}{t + \Delta t}$$

The second step is to obtain a formula for the change in position. This is done

by subtracting each member of the original equation from the corresponding member of the new equation.

$$\begin{aligned}\Delta s &= \frac{2 - (t + \Delta t)}{t + \Delta t} - \frac{2 - t}{t} \\ &= \frac{-2\Delta t}{t(t + \Delta t)}\end{aligned}$$

The third step is to divide both members by Δt .

$$\frac{\Delta s}{\Delta t} = -\frac{2}{t(t + \Delta t)}$$

The fourth and last step is to allow Δt to approach zero. The velocity is thus found to be

$$\begin{aligned}\frac{ds}{dt} &= \lim_{\Delta t \rightarrow 0} \left[-\frac{2}{t(t + \Delta t)} \right] \\ &= -\frac{2}{t^2}\end{aligned}$$

Exercises

Find the velocity at any time in each of the following cases.

1. $s = \frac{3}{2 - t}$

2. $s = \frac{2 + t}{2 - t}$

3. $s = at^2 + bt + c$

4. $s = \frac{k}{t}$

5. $s = kt^3$

6. $s = kt^4$

147. Rules for Differentiation. A limit of the form 21-5 is called the *derivative of s with respect to t* . The process of calculating derivatives is called differentiation. We proceed to show how derivatives may be obtained with the help of certain formulas.

The derivative of a constant is zero. It is evident physically that if a point remains fixed, its velocity is zero. In symbols, we may write

$$\frac{d(k)}{dt} = 0 \qquad 21-6$$

We next consider the effect of a constant multiplier. Set

$$s = ku$$

where u represents any function of t . During an interval Δt , the functions u and s will change by amounts Δu and Δs respectively. Then

$$s + \Delta s = k(u + \Delta u)$$

which, treated in the usual way, leads to

$$\Delta s = k \Delta u$$

Dividing both members by Δt , we have

$$\frac{\Delta s}{\Delta t} = k \frac{\Delta u}{\Delta t}$$

The limit of $\Delta u/\Delta t$ as Δt approaches zero is du/dt by definition. Hence

$$\frac{ds}{dt} = k \frac{du}{dt}$$

This result may also be written in the form

$$\frac{d(ku)}{dt} = k \frac{du}{dt} \quad 21-7$$

Symbolically, we may say that a constant multiplier passes the symbol of differentiation, which is d/dt .

The derivative of a sum is the sum of the derivatives. Let

$$s = u + v$$

where u and v represent any functions of the time. During an interval Δt , u and v change by amounts Δu and Δv , and therefore s changes by an amount Δs . We have then

$$s + \Delta s = (u + \Delta u) + (v + \Delta v)$$

whence

$$\Delta s = \Delta u + \Delta v$$

Dividing both members by Δt , the result is

$$\frac{\Delta s}{\Delta t} = \frac{\Delta u}{\Delta t} + \frac{\Delta v}{\Delta t}$$

As Δt approaches zero, the three average velocities approach instantaneous velocities. Accordingly,

$$\frac{ds}{dt} = \frac{du}{dt} + \frac{dv}{dt}$$

Or

$$\frac{d}{dt} (u + v) = \frac{du}{dt} + \frac{dv}{dt} \quad 21-8$$

Let us next consider the power function which was encountered in Chapter 17, taken in the form

$$s = t^n$$

It is easily verified that

$$\frac{d}{dt}(t^2) = 2t$$

$$\frac{d}{dt}(t^3) = 3t^2$$

$$\frac{d}{dt}(t^4) = 4t^3$$

$$\frac{d}{dt}(t^{-1}) = -\frac{1}{t^2}$$

and so on. These are all special cases of the formula

$$\frac{d}{dt}(t^n) = nt^{n-1} \qquad 21-9$$

That is, to obtain the derivative of a power of t , write the exponent as a coefficient of a power of t which is less by 1 than the original power. A satisfactory proof of the general formula is outside of the scope of this book, though it is readily verified in particular cases.

Example 1. Differentiate $11t^3$ by rule.

Applying the constant multiplier rule (equation 21-7), we have

$$\frac{d}{dt}(11t^3) = 11 \frac{d}{dt}(t^3)$$

Applying the power law (equation 21-8), the derivative is found to be $33t^2$.

Example 2. Differentiate $\frac{2-t}{t}$ by rule.

In order to use the rule for the derivative of a sum, we write

$$\frac{2-t}{t} = \frac{2}{t} - 1$$

In order to use the power rule, the first term is written with a negative exponent.

Then

$$\begin{aligned}\frac{d}{dt}\left(\frac{2-t}{t}\right) &= \frac{d}{dt}(2t^{-1} - 1) \\ &= 2(-1)t^{-2} \\ &= -\frac{2}{t^2}\end{aligned}$$

in agreement with a previous result (page 266).

Exercises

Differentiate by rule:

1. $at^2 + \frac{b}{t}$

2. $3t^{1.1} - 4t^{-1.1}$

3. $\frac{t^2 - 17t + 41}{2t}$

4. $a\sqrt{t} - \frac{b}{\sqrt{t}}$

5. $\sqrt{3t} - \sqrt{\frac{3}{t}}$

6. $\left(\frac{a}{t}\right)^{1/2} - \left(\frac{t}{a}\right)^{1/2}$

7. $\sqrt[3]{7at^2}$

8. $\sqrt{t(2t^2 - 3t^4)}$

148. The Derivative of a Product. Let

$$s = uv$$

where u and v are arbitrary functions of t . During the interval Δt , u and v change by amounts Δu and Δv , and s changes by an amount Δs . Therefore

$$\begin{aligned}s + \Delta s &= (u + \Delta u)(v + \Delta v) \\ &= uv + u\Delta v + v\Delta u + \Delta u\Delta v\end{aligned}$$

which reduces to

$$\Delta s = u\Delta v + v\Delta u + \Delta u\Delta v$$

Dividing both members by Δt , we have

$$\frac{\Delta s}{\Delta t} = u \frac{\Delta v}{\Delta t} + v \frac{\Delta u}{\Delta t} + \Delta u \left(\frac{\Delta v}{\Delta t}\right)$$

As Δt approaches zero, the average velocities approach the instantaneous velocities, and Δu approaches zero. Hence

$$\frac{ds}{dt} = u \frac{dv}{dt} + v \frac{du}{dt} \qquad 21-10$$

To find the derivative of the product of two factors, multiply the first factor by the derivative of the second, and add the second factor multiplied by the derivative of the first.

149. The Derivative of a Function of a Function. Suppose that s is a function of u , and u in turn is a function of t . During the interval Δt , u changes by an amount Δu and s by an amount Δs . The equation

$$\frac{\Delta s}{\Delta t} = \frac{\Delta s}{\Delta u} \frac{\Delta u}{\Delta t}$$

is an obvious identity. Now as Δt approaches zero, the three quotients each approach derivatives. We may therefore write

$$\frac{ds}{dt} = \frac{ds}{du} \frac{du}{dt} \quad 21-11$$

To illustrate the use of this important formula, suppose that we wish to differentiate

$$s = u^n$$

where, as usual, u is a function of t . Since the letters used to write our formulas are arbitrary, we have by the power formula

$$\frac{ds}{du} = nu^{n-1}$$

Now by formula 21-11

$$\frac{ds}{dt} = nu^{n-1} \frac{du}{dt}$$

This result may also be written

$$\frac{d}{dt}(u^n) = nu^{n-1} \frac{du}{dt} \quad 21-12$$

Example 1. Differentiate $\sqrt{t^2 - 12}$ by rule.

In this example, $u = t^2 - 12$, and $n = \frac{1}{2}$. Accordingly,

$$\begin{aligned} \frac{d}{dt}(t^2 - 12)^{1/2} &= \frac{1}{2}(t^2 - 12)^{-1/2} \frac{d}{dt}(t^2 - 12) \\ &= \frac{t}{\sqrt{t^2 - 12}} \end{aligned}$$

Example 2. Differentiate $\frac{a+t}{a-t}$ by rule.

Writing the quotient in the form of a product, we have

$$\begin{aligned}\frac{d}{dt} \left(\frac{a+t}{a-t} \right) &= \frac{d}{dt} [(a+t)(a-t)^{-1}] \\ &= (a+t) \frac{d}{dt} (a-t)^{-1} + (a-t)^{-1} \frac{d}{dt} (a+t) \\ &= (a+t)(-1)(a-t)^{-2}(-1) + (a-t)^{-1} \\ &= \frac{a+t}{(a-t)^2} + \frac{1}{a-t} \\ &= \frac{2a}{(a-t)^2}\end{aligned}$$

Exercises

Differentiate:

1. $\sqrt[3]{a+t}$

2. $\sqrt{\frac{a}{t} - \frac{t}{a}}$

3. $\frac{\sqrt{t}+2}{2-\sqrt{t}}$

4. $\left(1 + \frac{2t^3}{5}\right)^{1/2}$

5. $\left(\frac{3+2t}{3-2t}\right)^{3/2}$

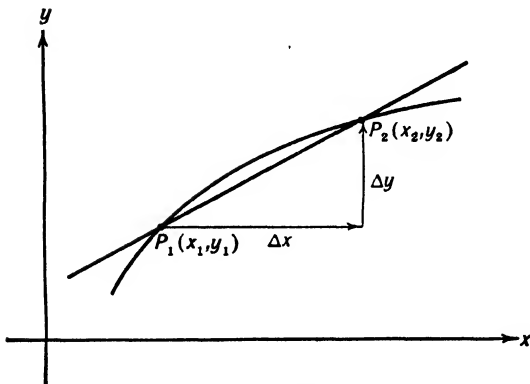


FIG. 138

150. Geometric Meaning of the Derivative. Consider the curve drawn in Figure 138. Let P_1 and P_2 be two points on the curve. The slope of the secant line drawn through P_1 and P_2 is given by the slope formula

$$\text{slope of secant} = \frac{y_2 - y_1}{x_2 - x_1}$$

It is convenient to express this in the delta notation. Accordingly, we have

$$\text{slope of secant} = \frac{\Delta y}{\Delta x} \Big|$$

If we hold the point P_1 fixed, and allow P_2 to move along the curve toward it, an interesting situation arises. For curves of the ordinary kind, what happens is that the secant line approaches a limiting position, represented by the line tangent to the curve at P_1 . Thus, as P_2 approaches P_1 , the slope of the secant line approaches the slope of the tangent line at P_1 .

As P_2 approaches P_1 , both Δx and Δy approach zero. The ratio $\Delta y/\Delta x$ approaches a limiting value, called the derivative of y with respect to x . We thus have

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \text{slope of tangent} \quad 21-13$$

The rules for differentiation that have been obtained are of course valid when y and x are written for s and t . ✓

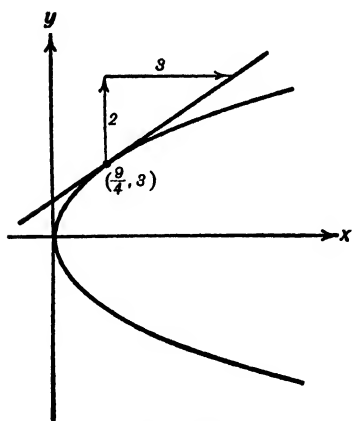


FIG. 139

Example. Find the slope of the parabola

$$y^2 = 4x$$

at the point $(\frac{9}{4}, 3)$.

Solving for y , we find

$$y = 2x^{1/2}$$

By the power formula,

$$\begin{aligned} \frac{dy}{dx} &= 2\left(\frac{1}{2}\right)x^{-1/2} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

Hence the required slope is $\frac{2}{3}$. The curve, and the tangent line, are drawn in Figure 139.

Exercises

Find the slope of each of the following curves at the given point. Sketch the curve and the tangent line.

1. $y = 2x^2$ at $(1, 2)$
2. $xy = 12$ at $(4, 3)$
3. $(y - 1)^2 = 2(x - 3)$ at $(5, -1)$
4. $y = \sqrt{16 - x^2}$ at $(2, 2\sqrt{3})$

5. Show that the slope of the normal to the curve $x^2 = ay$ at any point is $-a/2x$.
6. Show that the line joining any point $P(x, y)$ on the parabola $x^2 = ay$ with the focus, and the line through P parallel to the axis of the parabola, make equal angles with the normal to the parabola at P .

151. Implicit Differentiation. When an equation has been solved for y in terms of x , it is said to be in *explicit* form. Otherwise, y is said to be an *implicit* function of x . It is often awkward, and sometimes impossible, to express a given function explicitly, in order to calculate the derivative. Fortunately, the derivative may be calculated directly, by means of formula 21-11.

Example 1. Differentiate $x^2 + y^2 = r^2$.

The derivative of x^2 is evidently $2x$. However, since y is regarded as a function of x , we have (formula 21-12)

$$\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$$

Hence, taking the derivative of both members of the given equation, we have, since r is constant,

$$\begin{aligned} 2x + 2y \frac{dy}{dx} &= 0 \\ \therefore \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

Example 2. Differentiating

$$xy = c$$

we have

$$\begin{aligned} x \frac{dy}{dx} + y &= 0 \\ \therefore \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

Exercises

Find $\frac{dy}{dx}$:

1. $3x^2 + 4y^2 = 24$
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
3. $5x^2 - 20x - y^2 = 0$
4. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
5. $x^{1/2} + y^{1/2} = a^{1/2}$
6. $x^2 + xy + y^2 = 36$

7. Show that the slope of the normal to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at any point $P(x,y)$ is a^2y/b^2x .
8. Show that the two lines joining a point $P(x,y)$ on an ellipse with the focal points make equal angles with a normal to the ellipse at P .

152. Related Rates of Change. Consider the mechanism shown in Figure 140. A bar AB , 2 feet long, is constrained to move in such a way that point A is always on the y -axis, and point B is always on the x -axis. Let the velocity of B be constantly 3 feet per second to the right. We wish to find the velocity of A at any instant.

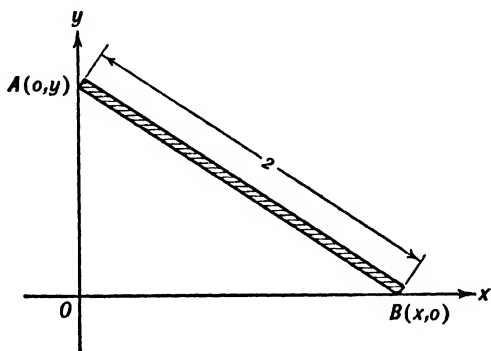


FIG. 140

The distances x and y are connected by the equation

$$x^2 + y^2 = 4$$

Let us take the derivative of both members with respect to the time.

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

Now $dx/dt = 3$, the velocity of B . Hence

$$\frac{dy}{dt} = -\frac{3x}{y}$$

is the required formula for the velocity of A . The negative sign shows that point A is moving downward.

We have seen that the derivative of distance with respect to time gives the velocity of a moving point. In general, the derivative is the *rate of change* of any quantity with respect to another. For example, suppose that

gasoline is entering a conical tank (Figure 141) at the rate of 80 cubic feet per minute. At what rate is the level of the gasoline rising when the depth is 15 feet?

Suppose that the proportions of the tank are such that the diameter equals $\frac{2}{3}$ of the height. Then the volume at any moment is given by the formula

$$V = \frac{\pi d^2 h}{12}$$

Setting $d = \frac{2}{3}h$, we may write

$$V = \frac{\pi h^3}{27}$$

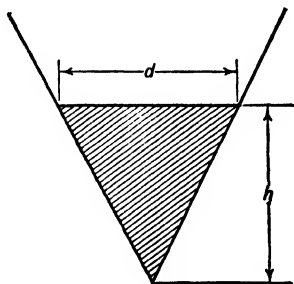


FIG. 141

Now the rate at which the volume of gasoline changes is 80 cubic feet per minute. That is

$$\frac{dV}{dt} = 80$$

But

$$\frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}$$

Hence, when $h = 15$, we have

$$\begin{aligned} \frac{dh}{dt} &= \frac{9(80)}{\pi(15)^2} \\ &= 1.02 \text{ feet per minute.} \end{aligned}$$

Exercises

1. Find the velocity of point A (Figure 140) when AB makes an angle of 20° with the horizontal.
2. The volume and pressure of a gas are related by the formula

$$pv = 30$$

where p is expressed in pounds per square inch, and v in cubic inches. If pressure is increasing at the rate of 5 pounds per square inch per minute, at what rate is the volume changing when $p = 6$ pounds per square inch?

3. A boat is pulled towards a dock 12 feet above the water level by means of a rope. If the rope is drawn in at the rate of 2 feet per second, how fast is the boat moving when it is 20 feet distant from the dock?

4. Figure 142 shows a slotted bar, free to turn about a smooth pin at B , with a block A which slides along the slot, remaining always on the x -axis. At a certain instant, $OA = 7$ inches, and is increasing at the rate of 2 inches per second. Find the rate at which block A is moving with respect to the slotted bar.

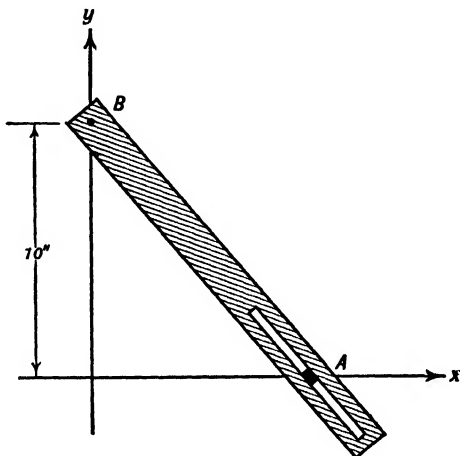


FIG. 142

153. Acceleration. The acceleration of a moving object bears the same relation to the velocity of the object, as the velocity does to the distance from a fixed point. The average acceleration during a time interval is defined to be the *change in velocity* divided by the *time during which the change occurs*. That is,

$$\text{ave. acc.} = \frac{\Delta v}{\Delta t} \quad 21-14$$

where v represents the velocity. The acceleration at any moment is defined to be the limit of the average acceleration, calculated during an interval which includes the moment in question, as the interval is taken shorter and shorter. That is, the acceleration at any moment is the derivative of the velocity with respect to the time.

$$\text{acceleration} = \frac{dv}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} \quad 21-15$$

154. Second Derivatives. Since velocity is the derivative of s , and acceleration is the derivative of the velocity, acceleration may be regarded as the derivative of a derivative. This is called a *second derivative*, and is written d^2s/dt^2 .

It is sometimes necessary to calculate derivatives of the third order, or of higher orders. In endeavoring to grasp the relationships between the derivatives of various orders, the student may find it helpful to consider a commuter riding home on a crowded bus at night. The quantity s may be taken to be the mileage registered by the speedometer. It is not perceived directly, but the commuter is aware of his position as he sees various familiar landmarks. The velocity, ds/dt , gives the speed of the bus. It is not perceived directly, but it can be estimated by the rate at which the lamp-posts appear to move past the bus. The acceleration, d^2s/dt^2 , is equal to the rate at which the speed is changing. The commuter perceives acceleration directly, for he must brace himself on one foot or the other as the driver alternately steps on the brake or gas pedal. The rate of change of acceleration, d^3s/dt^3 , is the factor which shakes up the passengers, as when the brakes are applied suddenly. In this illustration, the bus has been considered as traveling in a straight line, and the effect of curving the path of the bus is neglected.

Exercises

1. If $v = 0.30t^2 + \frac{0.50}{t}$, find the acceleration at any moment.

$$\text{Ans. } 0.60t - \frac{0.50}{t^2}$$

2. If $v^3 = t + k$, find the acceleration at any moment.

3. If $s = \frac{t-1}{t+1}$, find $\frac{d^2s}{dt^2}$.

$$\text{Ans. } -\frac{4}{(t+1)^3}$$

4. If $s^2 = t^2 + 5$, find $\frac{d^2s}{dt^2}$.

5. If $s = \frac{1}{4}t^5 + 3t$, find the rate at which the acceleration is changing when $t = 2$.

$$\text{Ans. } 60$$

6. If $s = 2(t-1)^4$, find the rate at which the acceleration is changing when $t = 3$.

155. Integration. In many problems, the acceleration at any time t is known, and it is required to find the velocity. This means finding a function whose derivative is known. The process is called *integration*, and the result is called the integral of the known function.

Suppose that the acceleration of a moving object is expressed by the equation

$$\frac{dv}{dt} = 3t^2$$

It can be seen that

$$v = t^3$$

is one formula for the velocity at any time. But there are others. For example, $v = t^3 + 20$ and $v = t^3 - 5$ both satisfy the acceleration equation. There is in fact an infinite set of velocity functions, all of which satisfy the given acceleration equation. It can be shown, however, that all of these velocity functions can be represented by the equation

$$v = t^3 + C$$

where the constant C is arbitrary. It is characteristic of the process of integration that an additive constant occurs in all cases. This constant is called the constant of integration. In order to determine its value in a given problem, additional information must be given.

Exercises

- Find the velocity function, if $\frac{dv}{dt} = 4t^3 - 5$, given that $v = 10$ when $t = 0$.
Ans. $v = t^4 - 5t + 10$
- Find the velocity function, if $\frac{dv}{dt} = 2t^2 - 3t$, given that $v = 12$ when $t = 0$.
- Find the velocity function, if $\frac{dv}{dt} = t - \frac{2}{t^3}$, given that $v = 0$ when $t = 1$.
Ans. $v = \frac{t^2}{2} + \frac{1}{t^2} - \frac{3}{2}$
- Find the velocity function, if $\frac{dv}{dt} = \frac{1 - t^2}{t^2}$, given that $v = \frac{1}{2}$ when $t = 2$.
- Find the distance traveled from $t = 0$ to $t = 6$, if $\frac{ds}{dt} = 3t^2 - t$. Ans. 198
- Find the distance traveled from $t = 0$ to $t = 5$, if $\frac{ds}{dt} = \frac{1}{2\sqrt{t}}$. (Observe that the constant of integration is zero in this instance.)

156. Integration Formulas. The sign of integration is \int . It is customary to indicate the variable with respect to which the integration is carried out, by writing it at the end of the function to be integrated, with the letter d attached as a prefix. Thus

$$\int 4t^3 dt$$

represents the function whose derivative with respect to t is $4t^3$.

Integration is an indirect, or tentative, process, and hence is more difficult than differentiation. It is usually carried out by comparing the unknown integral with a table of known integrals, with the object of

reducing the unknown form to a known one. A systematic exposition of the technique of integration is beyond the scope of this book, but certain standard formulas can be obtained directly from the formulas for differentiation.

From equation 21-7 we have

$$\int k u \, dt = k \int u \, dt \quad 21-16$$

A constant factor can be removed from within the sign of integration.

From equation 21-8, we have

$$\int (u + v) \, dt = \int u \, dt + \int v \, dt \quad 21-17$$

The integral of a sum is the sum of the integrals.

From equation 21-9 we have

$$\int t^n \, dt = \frac{t^{n+1}}{n+1} + C \quad 21-18$$

The integral of a power of a quantity is the quantity with the exponent increased by 1, divided by the exponent increased by 1. Formula 21-18 fails for $n = -1$. A discussion of this case is beyond the scope of this book.

Example.

$$\begin{aligned} \int \left(3t^2 + \frac{1}{t^2} \right) dt &= \int 3t^2 \, dt + \int \frac{dt}{t^2} \\ &= 3 \int t^2 \, dt + \int t^{-2} \, dt \\ &= 3 \left(\frac{t^3}{3} \right) + \frac{t^{-1}}{-1} + C \\ &= t^3 - \frac{1}{t} + C \end{aligned}$$

Exercises

Integrate the following, and check by finding the derivative of your result.

1. $\int \frac{t^5}{2} \, dt$
2. $\int \left(at^2 - \frac{b}{t^3} \right) dt$
3. $\int \frac{0.35 - t}{t^3} \, dt$
4. $\int (t - 2)^2 \, dt$
5. $\int 62.5t(20 - t^2) \, dt$
6. $\int \pi t^2(1 - t) \, dt$

157. The Area Under a Curve. An important application of integration is in the calculation of areas. Consider a curve whose equation is $y = f(x)$. Let us find a formula for the rate at which the area under the curve changes with respect to x (Figure 143). We shall use the general method of finding the average rate of change, and then calculating the limit of this average rate of change.

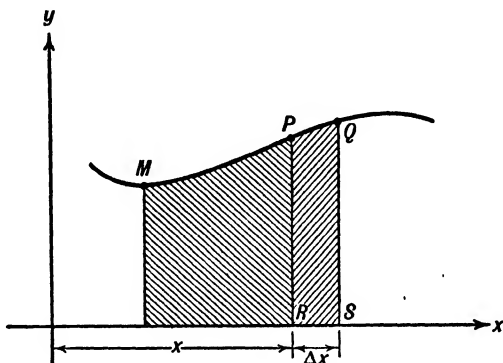


FIG. 143

In Figure 143, let A represent the area under the curve, between the ordinate at some fixed point M , and the ordinate at any other point $P(x, y)$. Let ΔA represent the change in A which corresponds to a change of Δx in x . It is evident, at least for a curve of the kind represented in Figure 143, that ΔA lies between the values $(PR)(\Delta x)$ and $(QS)(\Delta x)$; that is,

$$y \Delta x < \Delta A < (y + \Delta y) \Delta x$$

Now if unequals are divided by equals, the results are unequal in the same order. Hence

$$y < \frac{\Delta A}{\Delta x} < y + \Delta y$$

Now as the interval Δx is shortened, so that the point Q moves along the curve toward P , the ordinate $QS = y + \Delta y$ approaches the ordinate $PR = y$ in length. That is, the right-hand member of the last-written inequality approaches the value of the left-hand member as a limit. It follows that the middle member approaches the same limit; that is,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx} = y$$

The rate of change of area with respect to the abscissa at any point is equal in value to the ordinate at that point.

This result is usually expressed in integral notation, as follows:

$$\text{Area} = \int y \, dx \qquad 21-19$$

Example. Find the area under the parabola $y = x(2 - x)$ from $x = 0$ to $x = 2$.

We have, from formula 21-19,

$$\begin{aligned} A &= \int x(2 - x) \, dx \\ &= \int (2x - x^2) \, dx \\ &= x^2 - \frac{x^3}{3} + C \end{aligned}$$

To evaluate the constant of integration, we observe that $A = 0$ when $x = 0$, so that $C = 0$. Hence, when $x = 2$, we have

$$\begin{aligned} A &= 2^2 - \frac{2^3}{3} \\ &= \frac{4}{3} \end{aligned}$$

Exercises

Find the areas under each of the following curves, between the ordinates indicated. A figure should be drawn in each case.

1. $y = 2x + 1$, from $x = 0$ to $x = 4$. Ans. 20
2. $y = \frac{1}{2}x - 1$, from $x = 4$ to $x = 6$.
3. $y = 3x - x^2$, from $x = 0$ to $x = 3$. Ans. $\frac{9}{2}$
4. $y = 3x - x^2$, from $x = 1$ to $x = \frac{3}{2}$.
5. $y = \sqrt{x}$, from $x = 0$ to $x = 4$. Ans. $\frac{1}{3}^{\frac{2}{3}}$
6. $y = \sqrt{x}$, from $x = 1$ to $x = 9$.
7. $y = mx + b$, from $x = 0$ to $x = 1$. Ans. $b + m/2$
8. $y = x(a - x)$, from $x = 0$ to $x = a$.

CHAPTER 22

PERMUTATIONS AND COMBINATIONS

158. Introduction. There is an aspect of uncertainty about life as we experience it. An event may come to pass, or not; our knowledge is often insufficient to enable us to predict with certainty that it will happen, or that it will not happen. It would appear, at first thought, that a situation of this kind hardly lends itself to mathematical reasoning. Nevertheless, the mathematical theory of probability is a most powerful tool for attacking many problems of science and engineering. Moreover, the theory of probability has deeply colored the world-view of civilized Western man; our people have a different approach to life than the people of other cultures and other times, in part because some of the concepts associated with the theory of probability have shaped, to a degree, our everyday patterns of thought. For example, the practice of carrying insurance against the ordinary hazards of life is a unique feature of our culture.

159. The Number of Ways in Which Independent Events Can Occur. Suppose that two switchboards, A and B , are connected by four telephone lines, and that B is connected to a third switchboard, C , by five telephone lines. In how many different ways can A be connected with C ? Evidently the total number of ways is $(4)(5) = 20$.

In general, if a thing can be done in n_1 different ways, and another thing can then be done in n_2 different ways, the total number of ways in which the two things can be done in succession is

$$N = n_1 n_2 \qquad 22-1$$

160. The Permutations of n Different Objects. How many six-digit numbers can be written, using each of the digits 1, 2, 3, 4, 5, 6 once only? The first place in the number can be filled with any one of the six digits. The second place can then be filled with any one of the remaining five digits. Hence the first and second places can be filled in $(6)(5)$ ways altogether. The third place can now be filled with any one of the remaining four digits. Proceeding in this way, we find that the total is

$$6! = (6)(5)(4)(3)(2)(1)$$

The symbol $6!$ is read *six factorial*, and means the product of the positive

The Permutations of n Objects Not All Different 283

integers from 1 to 6 inclusive. Factorials occur frequently in the formulas of the theory of probability.

If we have n different objects, the number of arrangements that can be made from them, using all of the objects in each arrangement, is

$$n! = (1)(2)(3) \cdots (n) \quad 22-2$$

Next, let us suppose that we have n different objects, and wish to compute the total number of arrangements, using exactly r of the objects in each arrangement. The first selection may be made in n different ways; the second selection may be made in $n - 1$ different ways; and we may proceed in this way, until r selections have been made. The total number of arrangements is

$${}_nP_r = (n)(n-1)(n-2) \cdots (n-r+1) \quad 22-3$$

The symbol ${}_nP_r$ represents the number of permutations (or arrangements) of n objects, of which no two are alike, that can be formed by taking r at a time.

If the right-hand member of equation 22-3 be multiplied and divided by $(n-r)!$, the convenient formula

$${}_nP_r = \frac{n!}{(n-r)!} \quad 22-4$$

is obtained.

161. The Permutations of n Objects Not All Different. How many different arrangements can be made, using the six letters of the word *banana*? Considering any particular arrangement, it is clear that permuting the a 's among themselves will not give a different arrangement; nor will interchanging the n 's result in a new arrangement. But the a 's can be permuted among themselves in $3!$ ways, and the n 's in $2!$ ways. The total number of permutations of six letters is $6!$. This is equal to the number of *different* arrangements, multiplied by $3!2!$. That is, the number of different arrangements is

$$\frac{6!}{3!2!} = 60$$

In the general case, if n objects are taken all at a time, there being n_1 of one kind, n_2 of another kind, n_3 of a third kind, and so on, the number of distinguishable permutations is

$$\frac{n!}{n_1! n_2! n_3! \cdots} \quad 22-5$$

162. The Combinations of n Different Objects. An arrangement of a set of objects is called a permutation; the set itself, independent of the arrangement or order, is called a *combination*. Hence in a single combination there may be many permutations.

Let us calculate the number of different combinations, each containing just 13 cards, that can be constructed from a deck of 52 different cards. A bridge hand is an example of such a combination; for any particular hand, the various possible arrangements are of no consequence. Now the cards in any such hand can be permuted among themselves in $13!$ ways. The total number of permutations is equal to the number of combinations, multiplied by $13!$ In symbols,

$${}_{52}P_{13} = {}_{52}C_{13}(13!)$$

That is,

$${}_{52}C_{13} = \frac{52!}{13!39!}$$

This works out to about $6(10)^{11}$, so that even an enthusiast has small prospect of holding a particular hand more than once in a lifetime, in ordinary play.

In general, the number of different combinations of n objects, of which no two are alike, that can be formed, taking just r objects each time, is

$${}_nC_r = \frac{n!}{(n-r)!r!} \quad \text{22-6}$$

Exercises

- How many different arrangements can be made from the eight letters of the word *formulas*?
- How many different arrangements can be made from the twelve letters of the word *nevertheless*?
- How many four-digit numbers can be formed, using digits from 1 to 9, without using any digit more than once?
- Is it true in general that $(n-r)! = n! - r!$?
- Using the digits 1, 2, 3, 4, write out the 24 three digit numbers that can be formed from them. Are these a set of permutations or of combinations?
- Find the values of ${}_{10}C_2$ and ${}_{10}C_3$; of ${}_{10}C_3$ and ${}_{10}C_7$. Prove that ${}_nC_r = {}_nC_{n-r}$.
- Through eleven points on a circle, how many different straight lines can be drawn?
Ans. 55
- If five points are known, no three of which lie on the same straight line, how many triangles can be formed by joining three of the points?
- Write out and simplify ${}_nC_1$ and ${}_nC_{n-1}$; ${}_nC_2$ and ${}_nC_{n-2}$.
- What is the value of ${}_nC_n$? Note that the formula yields a result concordant with the meaning, only if $0!$ is taken to be 1.
- How many telephone lines are needed to connect each of four offices directly to all of the others?

163. The Binomial Theorem. Consider the expression

$$(x + y)^3 = (x + y)(x + y)(x + y)$$

In order to expand the right-hand member, it is necessary to take the sum of all possible products, each containing one and only one term from each factor. We may take the first term from each factor, which can be done in only one way; the product is x^3 . Next, we may take the x 's from any two factors, and the y from the third factor. This can be done in ${}_3C_1 = 3$ ways. The sum of the products is therefore $3x^2y$. Proceeding in this way, we have

$$\begin{aligned}(x + y)^3 &= x^3 + {}_3C_1x^2y + {}_3C_2xy^2 + y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

Next, let us consider the expression

$$(x + y)^n = (x + y)(x + y)(x + y) \cdots (x + y)$$

where n is a positive integer. The right-hand member contains just n factors. In expanding it, we take the sum of all possible products, each containing one and only one term from each factor. If we take the first term from each factor, which can be done in just one way, the product is x^n . If we take the x 's from every factor except one, and the y from that factor, which can be done in ${}_nC_1$ ways, the sum of these products will be ${}_nC_1x^{n-1}y$. If we take the x 's from $n - r$ factors, which can be done in ${}_nC_r$ ways, the sum of the products will be ${}_nC_rx^{n-r}y^r$. Hence

$$(x + y)^n = x^n + {}_nC_1x^{n-1}y + \cdots + {}_nC_nx^{n-n}y^n \quad 22-7$$

This formula has the advantage of enabling us to write down *any desired term* in the expansion of $(x + y)^n$, without actual multiplication, and without calculating other terms. The C 's are called the *binomial coefficients*.

By using the factorial expressions for the C 's (equation 22-6), the binomial expansion 22-7 may be put in the form

$$\begin{aligned}(x + y)^n &= x^n + \frac{n}{1}x^{n-1}y + \frac{n(n-1)}{1(2)}x^{n-2}y^2 \\ &\quad + \frac{n(n-1)(n-2)}{1(2)(3)}x^{n-3}y^3 + \cdots \quad 22-8\end{aligned}$$

Observe that the exponent of x is diminished by 1 in successive terms, while the exponent of y is increased by 1. *If the coefficient of any term is multiplied by the power of x in the same term, and divided by the number of the term, the result is the coefficient of the succeeding term.*

Formula 22-8 has been established only when n is a positive integer. In this case there are evidently $n + 1$ terms in the right-hand member. When n is any number other than a positive integer, formula 22-8 has no last term; the right-hand member becomes an infinite series, analogous to the geometric series of Chapter 15. With the help of the geometric series, it may be shown that the binomial series is divergent if x is less in absolute value than y , and convergent if x is greater in absolute value than y . It may also be shown that, in the latter case, the series converges to $(x + y)^n$, and may be used to represent it. The proofs of these statements are beyond the scope of this book.

Example. Find the first four terms in the binomial expansion of $(a^2 - 3b)^{11}$.

There is less chance of error if the work is done in two steps. First, taking $n = 11$, we have from formula 22-8

$$\begin{aligned}(x + y)^{11} &= x^{11} + 11x^{10}y + \frac{11(10)}{2}x^9y^2 + \frac{11(10)(9)}{2(3)}x^8y^3 + \dots \\ &= x^{11} + 11x^{10}y + 55x^9y^2 + 165x^8y^3 + \dots\end{aligned}$$

Now replace x by a^2 , and y by $-3b$. We have

$$\begin{aligned}(a^2 - 3b)^{11} &= (a^2)^{11} + 11(a^2)^{10}(-3b) + 55(a^2)^9(-3b)^2 + 165(a^2)^8(-3b)^3 + \dots \\ &= a^{22} - 33a^{20}b + 495a^{18}b^2 - 4455a^{16}b^3 + \dots\end{aligned}$$

Exercises

1. Write formula 22-7 in the sigma notation. (Observe that ${}_nC_0 = {}_nC_n = 1$ if 0! is defined to be equal to 1.)
2. Expand $(x + y)^4$ by direct use of combinational reasoning.
3. Find the first three terms in the expansion of $(2a + 7b)^8$.
4. Find the first three terms in the expansion of $(s^2 - 2t)^9$.
5. Find the first two terms in the expansion of $(x^2 + y)$. Use your result to compute the approximate value of $\sqrt{65}$ ($= \sqrt{64 + 1}$), $\sqrt{80}$, and $\sqrt{97}$.
6. Find the first three terms in the expansion of $(x + y)^{-1}$. Use your result to compute the approximate value of $\frac{1}{102} \left(= \frac{1}{100 + 2} \right)$; and of $\frac{1}{11}$.
7. Find the sixth term in the expansion of $(x + y)^{17}$; of $(bx - 2k)^{17}$.
8. Find the eighth term in the expansion of $(a^2 - bm)^{10}$.
9. Prove that the total number of combinations of n objects is $2^n - 1$; that is,

$${}_nC_1 + \dots + {}_nC_n = 2^n - 1$$

10. Express the formula of Exercise 9 in the sigma notation.

164. The Probability of an Event. If a card is drawn at random from a deck of 52, what are the chances that it will prove to be the ace of spades?

Evidently the chance is 1 in 52. An equivalent statement is that the probability of drawing the ace of spades is $\frac{1}{52}$.

Suppose that an event can happen in a ways, or fail in b ways; and that each of these $a + b$ ways is equally likely. Then the probability that the event will happen is defined to be

$$p = \frac{a}{a + b} \quad 22-9$$

That is, the probability of an event is the *number of favorable cases* divided by the *total number of cases*, favorable and unfavorable, provided that all cases are equally likely.

The probability that an event will fail is defined to be

$$q = \frac{b}{a + b} \quad 22-10$$

Adding the corresponding members of 22-9 and 22-10, we have

$$p + q = 1 \quad 22-11$$

That is, if the probability that an event will happen is p , the probability that it will fail is $1 - p$.

Example. A handful of n coins is tossed in the air. What is the probability of just r heads?

The number of favorable cases is ${}_nC_r$. The total number of cases is

$${}_nC_0 + {}nC_1 + {}nC_2 + {}nC_3 + \cdots + {}nC_n = 2^n$$

a result readily obtained from formula 22-7, by setting $x = y = 1$. Hence the probability is

$$\frac{{}_nC_r}{2^n}$$

It should be observed that the probabilities of 0, 1, 2, 3, \cdots heads are proportional to the binomial coefficients.

Exercises

1. If a coin is tossed in the air, what is the probability of heads?
2. If a die, with faces numbered from 1 to 6, is cast, what is the probability of throwing a 4?
3. Three coins are tossed. What is the probability of three heads? Of just two heads?
4. Two dice are thrown. What is the probability of a double six?
5. If an event is certain to happen, what is its probability?
6. If an event is certain to fail, what is its probability?

165. Independent Events. Two events are said to be independent when the happening or failing of one has no effect upon the probability of the other. Suppose that a deck of cards is examined, and found to contain the usual 52 cards, all different. One is drawn at random, and proves to be the ace of hearts. This card is then replaced, and the deck reshuffled. A second card is drawn, and proves to be the queen of spades. These two events are independent.

If the probability of the first of two independent events is p_1 , and of the second p_2 , the probability that both will happen is p_1p_2 . For, let

$$p_1 = \frac{a_1}{a_1 + b_1}$$

and

$$p_2 = \frac{a_2}{a_2 + b_2}$$

There are a_1a_2 favorable ways for both to happen, and $(a_1 + b_1)(a_2 + b_2)$ is the total number of ways in which either or both may happen or fail. Then the probability that both will happen is

$$\frac{a_1a_2}{(a_1 + b_1)(a_2 + b_2)} = p_1p_2$$

as was to be proved.

Example. The probability that the ace of hearts and the queen of spades will be drawn in that order, under the conditions given above, is

$$\left(\frac{1}{52}\right)\left(\frac{1}{52}\right)$$

166. Dependent Events. If the probability of an event is p_1 , and if, after this event has happened, the probability of a second event is p_2 , the probability that both events will occur in the order stated is p_1p_2 . The reasoning is similar to that used in establishing the theorem on independent events.

For example, suppose that two cards are drawn in succession from a deck of 52, the first *not* being replaced before the second is drawn. What is the probability that the first will be the ace of hearts, and the second the queen of spades?

The probability that the first card drawn will prove to be the ace of hearts is $\frac{1}{52}$. There remain 51 cards. When the second is drawn, the probability that it will prove to be the queen of spades is $\frac{1}{51}$. Hence the probability of drawing the ace of hearts, followed by the queen of spades, is

$$\left(\frac{1}{52}\right)\left(\frac{1}{51}\right)$$

167. Mutually Exclusive Events. Suppose that, if one event happens, a second event must fail; while, if the second event happens, the first must fail. The probability that *one or the other* will happen is the sum of the probabilities of each.

For, let the number of ways in which the first can happen be a_1 , and the number of ways in which the second can happen a_2 . Let N be the total number of ways in which either can happen, or both fail. Then

$$p_1 = \frac{a_1}{N}$$

and

$$p_2 = \frac{a_2}{N}$$

The number of ways in which either the first or the second can happen is $a_1 + a_2$. Hence the probability of one or the other happening is

$$\frac{a_1 + a_2}{N} = p_1 + p_2$$

as was to be proved.

Example. Let two cards be drawn at random from a deck of 52. What is the probability that they will prove to be the ace of hearts and the queen of spades?

The probability that the two cards will be drawn in the given order was found to be $(\frac{1}{52})(\frac{1}{51})$. The probability that the same cards will be drawn, but in reverse order, is also $(\frac{1}{52})(\frac{1}{51})$. The probability that either one of these mutually exclusive events will happen is

$$\frac{2}{(52)(51)}$$

As a check, let us calculate this probability directly. The number of two-card combinations that can be drawn is ${}_{52}C_2$. The number of favorable combinations is just 1. Hence the probability is

$$\frac{1}{{}_{52}C_2} = \frac{2}{(52)(51)}$$

as before.

Exercises

1. What is the probability that two kings will be drawn in succession from a deck of cards?
2. What is the probability that 4 cards, dealt from a deck of 52, will all be of the same suit?

3. The probability that A will be alive 10 years from now is $\frac{4}{7}$. The probability that B will be alive is $\frac{3}{4}$. What is the probability that both will be dead? A alive, and B dead? B alive, and A dead?
4. If two dice are thrown, what is the probability of either 7 or 11?
5. An urn contains 10 black and 6 white balls. What is the probability of drawing 2 black balls in succession? What is the probability of drawing a white ball, followed by a black one?
6. An urn contains 4 black and 4 white balls. A and B draw a ball in turn, until one or the other draws a white ball, and so wins the game. What is the probability that the first player will win?
7. Four balls are divided at random among three compartments. What is the probability that a particular compartment will contain just 2 balls?

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